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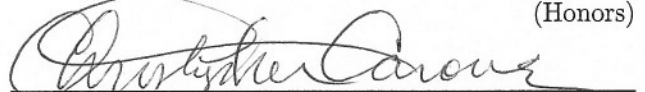
A G_2 Electroweak Model¹

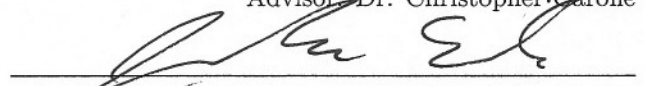
A thesis submitted in partial fulfillment of the requirement
for the degree of Bachelor of Science with Honors in
Physics from the College of William and Mary in Virginia,

by

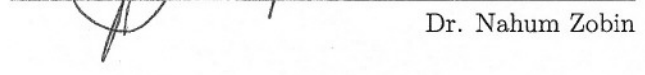
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Abstract

The minimal $SU(3)$ model for electroweak unification has been proposed and analyzed in the literature (see [3], [4]). The main attraction of this model is its prediction of $\sin^2 \theta_W = 1/4$, which holds approximately at 4 TeV. In this project we are interested in developing an extension to the minimal $SU(3)$ model, by embedding it in the exceptional Lie group G_2 . We begin by deriving various group theoretic properties and structures related to the group G_2 and its Lie algebra. We focus on aspects that could be useful for particle physics and extensions to the Standard Model. We discuss the minimal $SU(3)$ model in detail, and produce two extensions to it. Finally, we obtain constraints to the new physics from these models based on precision electroweak data.

Acknowledgements

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Chapter 1

Background and Motivation

1.1 Unified Theories and Electroweak Unification

A unified theory is a model that encompasses all the interactions of elementary particles within a single theoretical construct. The search for a successful unified theory is one of the primary outstanding issues in particle physics. In the 1970s, the first and simplest unified theory, the Georgi-Glashow model, was introduced [5]. Since then, a variety of similar theories have been proposed, and the subject has been vigorously explored.

Symmetry laws govern the interactions between elementary particles in almost all contemporary models. The Standard Model encodes three distinct symmetries, each associated with one of the three fundamental forces. Although the Standard Model has been extremely successful at explaining experimental observations, theories beyond the Standard Model have been an area of active research. Unified theories are a prominent group of extensions to the Standard model, in which the three fundamental forces are described in terms of a single symmetry.

The formulation of unified theories relies heavily upon group theory, which provides a mathematical framework for the study of symmetries. In group theory, symmetries are classified by mathematical groups of transformations. Lie groups and algebras are of particular significance, because they generate the types of continuous symmetries that arise in particle physics [6]. In the Standard Model, the electromagnetic force is associated with a $U(1)$ symmetry, which defines transformations by one-dimensional unitary matrices, i.e. complex phase rotations. The weak and strong forces arise from $SU(2)$ and $SU(3)$ symmetries respectively (the $SU(N)$ symmetries describe transformations by N -dimensional unitary matrices with unit determinants). The Georgi-Glashow model proposed a unified $SU(5)$ symmetry, into which all three Standard Model symmetries are embedded [5].

When introducing a symmetry into a theory of physics, we require that observable quantities be invariant under the symmetry transformations. In quantum field theory, a symmetry requires that the Lagrangian must be invariant under the relevant transformations. The results are generally very different for global versus local symmetries. A global symmetry transforms a given field in the same way at all points in space-time. In contrast, a local

symmetry acts independently at different points in space-time. For many global symmetries, it is straightforward to make the Lagrangian invariant. However, for a local symmetry, additional coordinate-dependent terms must be added to impose the desired invariance. The new terms are included in covariant derivatives, which take the place of the standard derivatives appearing in the original Lagrangian. The new terms correspond to the introduction of some force-carrying particle, known as a gauge boson. For example, when we impose the $U(1)$ symmetry, we must define the four-vector potential of electromagnetism in the new terms. In this case, the photon is the gauge boson that mediates the resulting electromagnetic force [7]. The local symmetry thereby generates a physical force and force-carrying particles.

Four fundamental forces are known to exist in nature: the electromagnetic force, the weak nuclear force, the strong nuclear force, and gravity. The Standard Model and unified theories are concerned with the first three (the inclusion of gravity is considered in other theories of particle physics, particularly string theory). The relative strengths of these forces are quantified by coupling constants. The hypothesis of force unification is empirically motivated by an observation that the coupling constants of the three forces appear to converge at a high energy scale (approximately 2×10^{16} GeV). The convergence suggests that at this scale, the three forces may be unified within a single symmetry.

Alternatively, one can consider electroweak unification, the unification of the electromagnetic and weak forces (or, more precisely, the $SU(2)$ and $U(1)$ gauge groups) in a symmetry separate from the strong force. The hypothesis of electroweak unification is motivated by its accurate prediction of the ratio of the $U(1)$ and $SU(2)$ coupling constants. The coupling constant ratio is measured via observation of the weak mixing angle, θ_W , which is defined in terms of these coupling constants as

$$\sin^2 \theta_W = \frac{g'^2}{g^2 + g'^2} \quad (1.1)$$

where the $U(1)$ and $SU(2)$ coupling constants are denoted by g' and g , respectively. When unified in $SU(3)$, for example, one obtains the two familiar coupling constants in terms of g_3 , the coupling constant of the $SU(3)$. It has been shown [2] that this yields

$$g = g_3, \quad g' = \frac{g_3}{\sqrt{3}}, \quad \text{and} \quad \sin^2 \theta_W = 1/4. \quad (1.2)$$

This final quantity, $\sin^2 \theta_W$ has been empirically measured in the rate of weak decays and scattering cross sections at CERN, with a value of 0.23120 [8]. The proximity of the empirical value to the prediction above indicates that an $SU(3)$ unification could occur at relatively low energy, on the order of 1-10 TeV [3].

1.2 Project Outline

In this project, we propose and explore a model for electroweak unification based on the group G_2 . Several factors inform our choice of gauge group. First, G_2 is of rank two; the rank two groups are the smallest groups that can contain the $SU(2)$ and $U(1)$ standard model groups,

and will therefore yield the simplest models. Additionally, G_2 contains as a subgroup the $SU(3)$ group discussed above. We can therefore carry over the successful weak mixing angle prediction for electroweak $SU(3)$ unification to our new G_2 model. Furthermore, extensions of the Standard Model based on the gauge group G_2 have not been widely studied; this approach may therefore yield novel results [9].

Our use of an unconventional symmetry group will require some preliminary work on the Lie theory of G_2 . For example, we must construct generators and representations of G_2 using some results of Lie theory [6]. We then follow the procedure for introducing a symmetry outlined above, using the new group. When imposing a local G_2 symmetry, additional terms will be introduced in covariant derivatives to produce an invariant Lagrangian. The new terms will contain information about the unified force and gauge bosons. Matter particles will be introduced by extending the gauge group in a way that preserves the weak mixing angle prediction. The complete model can be analyzed to explore predictions associated with the new electroweak unification.

We will perform a precision electroweak fit based on the electroweak observables to compare the resulting theory to experimental results. A statistical analysis will allow us to compute confidence contours in parameter space and designate regions that are excluded or physically allowable; we plan to compare these results to similar analyses for $SU(3)$ models.

We expect that the new model will produce a variety of observable predictions. The unified symmetry will predict the existence of new force-carrying particles, whose characteristics can be extracted from the model. We will be especially interested in developing predictions for the production, decay processes, and interactions of the new force carriers. From the statistical analysis, we will also be interested in computing the minimal allowed unification scale and exotic boson mass scale. A relatively light (1-2 TeV) exotic boson may allow us to develop interesting predictions relevant to near-term collider experiments. These and other related predictions would produce a variety of observable phenomena relevant to experiments at high-energy particle accelerators, particularly the Large Hadron Collider at CERN.

Chapter 2

$SU(3)$ Unification

In this chapter we will discuss the model for $SU(3)$ electroweak unification proposed in [3]. After we develop some group theoretic techniques, we will return to fill in complete details on this model and extend it with G_2 .

2.1 Notation for $SU(N)$ Groups

2.1.1 $SU(2)$ Group

$SU(2)$ is the group of 2×2 matrices with unit determinant. Its generators in the fundamental representation are given by $\tilde{t}^a = \frac{1}{2}\sigma_a$ where σ_a are the Pauli matrices. For reference, we list the Pauli matrices below.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The $SU(2)$ generators satisfy the standard orthonormality condition

$$\langle \tilde{t}^a | \tilde{t}^b \rangle = \text{Tr}(\tilde{t}^{a\dagger} \tilde{t}^b) = \frac{1}{2} \delta_{ab}.$$

An arbitrary element of $SU(2)$ is given by $e^{\chi_a \tilde{t}^a}$.

2.1.2 $SU(3)$ Group

$SU(3)$ is the group of 3×3 matrices with unit determinant. Its generators in the fundamental representation are given by $t^a = \frac{1}{2}\lambda_a$ where λ_a are the Gell-Mann matrices. For reference, we list the Gell-Mann matrices here:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The $SU(3)$ generators satisfy the standard orthonormality condition

$$\langle t^a | t^b \rangle = \text{Tr}(t^{a\dagger} t^b) = \frac{1}{2} \delta_{ab}.$$

An arbitrary element of $SU(3)$ is given by $e^{\chi_a t^a}$. $SU(3)$ contains as a maximal subgroup $SU(2) \times U(1)$. The generators $t^{1\dots 3}$ form an $SU(2)$ subgroup, since the Pauli matrices are easily identified in the upper 2×2 block. The $U(1)$ generator must commute with the $SU(2)$ subgroup, and is therefore given by t^8 , which has an identity matrix in its upper 2×2 block.

2.2 Weak Mixing Angle Prediction

The $SU(3)$ electroweak unification model is primarily attractive due to its prediction of the weak mixing angle, θ_W . The weak mixing angle is defined in terms of coupling constants as in Eq. (1.1); it gives the mixing between the Z boson and the photon. The model was introduced by Weinberg in [2].

In this model, we embed the Standard Model electroweak gauge group, $SU(2)_W \times U(1)_Y$ in $SU(3)$. We identify $SU(2)_W \times U(1)_Y$ as the subgroup described above. The relative normalization of the $SU(2)_W$, $U(1)_Y$ subgroups allows us to obtain the Standard Model coupling constants in terms of g_3 , the $SU(3)$ coupling constant. Consider putting Standard Model leptons in a $\mathbf{3}$ of $SU(3)$ as follows

$$\begin{pmatrix} (e_L)^c \\ -(\nu_L)^c \\ e_R \end{pmatrix} \sim \mathbf{3};$$

the superscript c represents charge conjugation, so that $(e_L)^c$ and $-(\nu_L)^c$ have hypercharge $Y = 1/2$, e_R has hypercharge $Y = -1$, and $\begin{pmatrix} (e_L)^c \\ -(\nu_L)^c \end{pmatrix}$ forms an $SU(2)$ doublet. This fixes the hypercharge operator from $U(1)_Y$ to be

$$Y = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \sqrt{3} t^8.$$

Since Y must be proportional to the t^8 generator from the $U(1)$ subgroup of $SU(3)$, we have

$$g_3 t^8 = g' Y.$$

Thus, $g' = \frac{g_3}{\sqrt{3}}$. Since the $SU(2)$ generators in $SU(3)$ are properly normalized to act on the doublet, we also have $g = g_3$, and thus we conclude that

$$\sin^2 \theta_W = \frac{g_3^2/3}{g_3^2 + g_3^2/3} = \frac{1}{4}.$$

Thereby we obtain the weak mixing angle prediction; its proximity to measured values indicates that such a unification could occur at the TeV scale.

2.3 $SU(3) \times SU(2) \times U(1)$ Unification

The immediate problem with an $SU(3)$ unification as proposed above is that there is no way to include the Standard Model quarks in the theory. One attractive solution proposed in [3] is to consider the alternative gauge group

$$SU(3) \times SU(2) \times U(1),$$

with couplings $g_3, \tilde{g}, \tilde{g}'$ respectively. For definiteness, let $SU(2)_0$ and $U(1)_0$ denote the respective subgroups of $SU(3)$ discussed above. All Standard Model fields may then be included in their usual representations, which transform only under the additional $SU(2) \times U(1)$ group. The Standard Model gauge group $SU(2)_W \times U(1)_Y$ is identified as the diagonal subgroup of $SU(2)_0 \times U(1)_0$ and the additional $SU(2) \times U(1)$. From the symmetry breaking, we can obtain the Standard Model couplings g, g' (corresponding to $SU(2)_W, U(1)_Y$ respectively) in terms of the gauge couplings $g_3, \tilde{g}, \tilde{g}'$. It will be shown in Ch. 6 that this yields

$$\frac{1}{g^2} = \frac{1}{g_3^2} + \frac{1}{\tilde{g}^2}, \tag{2.1}$$

$$\frac{1}{g'^2} = \frac{3}{g_3^2} + \frac{1}{\tilde{g}'^2}. \tag{2.2}$$

This result indicates that the $\sin^2(\theta_W)$ prediction from the $SU(3)$ model carries over in the limit that $\tilde{g}, \tilde{g}' \gg g_3$. Therefore we can work in this limit and maintain the successful weak mixing angle prediction.

Chapter 3

Roots and Weights

In this chapter we will introduce basic structures of Lie groups that allow us to construct any representation of any simple Lie group. The construction follows Ref. [6], Ch. 6, 8. Our approach relies on many useful properties of roots and weights, which are described and proved in [6].

To construct a particular representation, we will compute its generators, T^a , which form a representation of the corresponding Lie algebra and yield the group representation via

$$D(g(\chi^a)) = e^{i\chi^a T^a}$$

where D is the representation, and $g(\chi^a)$ is an arbitrary group element parameterized by χ^a . For some particular representation of the algebra, the largest commuting subalgebra is denoted the Cartan subalgebra. These generators can be simultaneously diagonalized, and so we can consider simultaneous eigenstates of the Cartan subalgebra. The weights of a representation are the eigenvalues of the Cartan subalgebra; weights are vectors with length equal to the cardinality of the Cartan subalgebra. The roots of a group are the weights of its adjoint representation. The simple roots are a set of roots that span the space of roots such that all other roots are integer linear combinations of these. As it turns out, the simple root structure completely defines the form of the complete group.

3.1 Dynkin Diagram and Simple Roots

Dynkin diagrams encode the simple roots for a particular Lie group, and provide all the information necessary to construct the group, algebra and respective representations. We will use the Dynkin diagram as a starting point for analyzing the group G_2 . The nodes of the diagram represent the simple roots, the arcs connecting nodes indicate angles between the respective roots, and arrows along the arcs indicate relative lengths of the root vectors. The root vectors for a group of rank N are elements of \mathbb{R}^N ; a proof in Ref. [6] Sec. 8.2.5 shows that a rank N simple group also has exactly N simple roots, which form a linearly independent basis for \mathbb{R}^N . We begin with the Dynkin diagram for G_2 , given in Fig 3.1.

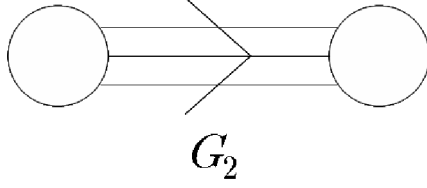


Figure 3.1: Dynkin diagram for group G_2 . The two nodes indicate the two simple roots of the group. The triple line connecting them indicates that they are at an angle of $\theta = \frac{5\pi}{6}$. The arrow points to the larger of the two, though in this case it is irrelevant.

The Dynkin diagram tells us that there are two simple roots, with an angle of $\theta = \frac{5\pi}{6}$ between them. We may fix the simple roots as

$$\alpha^1 = (0, 1) \quad (3.1)$$

$$\alpha^2 = \left(\frac{\sqrt{3}}{2}, -\frac{3}{2} \right). \quad (3.2)$$

Note that these choices are not unique. We are free to pick the location of the first root, and its normalization, and can then derive the rest of the roots from the properties of Lie groups, as described in [6]¹. After arbitrarily setting the first root α^1 , the direction of the second root is fixed due to the angle constraint between them; while there are two possibilities (taking angles clockwise or counter-clockwise), the result is identical up to reflections and rotations. It remains only to determine the relative normalizations. We then obtain

$$\frac{2\alpha^1 \cdot \alpha^2}{\alpha^{1^2}} = -n, \quad (3.3)$$

$$\frac{2\alpha^2 \cdot \alpha^2}{\alpha^{2^2}} = -m, \quad (3.4)$$

where both m, n are nonnegative integers. Multiplying these equations, we obtain

$$nm = \left(\frac{2\alpha^1 \cdot \alpha^2}{\alpha^1 \alpha^2} \right)^2. \quad (3.5)$$

These conditions give

$$nm = 4 \cos^2 \theta = k,$$

so that n is a divisor of the nonnegative integer² $k = 4 \cos^2 \theta$. However, k takes only values $\{0, 1, 2, 3\}$, since these are the only values of $4 \cos^2 \theta$ that are integers. So n has two possible values³, either $n = 1$ or $n = k$. This yields

$$\frac{|\alpha^2|}{|\alpha^1|} = \frac{n}{\sqrt{k}} \in \left\{ \sqrt{k}, \frac{1}{\sqrt{k}} \right\}. \quad (3.6)$$

¹See Ref. [6], 6.6 for notation and discussion of this procedure for general roots and weights; see Ref. [6] 8.2-8.4 for simple roots and Dynkin diagrams.

²We obtain these properties of the value of $k = 4 \cos^2 \theta$ from the ‘master formulas’ 6.36 and 6.39 in Ref. [6].

³For $k = 1$ the only value is $n = 1$. For $k = 0$, n may be any integer - in this case, the simple roots are orthogonal and the Lie group therefore non-simple. Here we are interested only in simple Lie groups. The procedure can be adapted to semi-simple Lie groups by identifying simple subgroups and applying this procedure.

The remaining ambiguity is resolved by the convention that an arrow is drawn from longer to shorter roots when $k \in 2, 3$. In our example, we had

$$\frac{|\alpha^2|}{|\alpha^1|} = \sqrt{k} = \sqrt{3}.$$

Consequently, the second root is uniquely determined by the Dynkin diagram, after the arbitrary choice of first root is made. The procedure can be applied inductively, so that for any number N of simple roots, the Dynkin diagram uniquely determines all roots up to the arbitrary choice of first root. At each step (say, the k -th root), the Dynkin diagram gives the angles between the next root and all the previous roots. Since the simple roots are linearly independent, this specifies the direction of the new k -th root in \mathbb{R}^k . The normalization is then obtained by the process above, completely fixing the new root in \mathbb{R}^k . Continuing in this manner will produce all simple roots in \mathbb{R}^N . So, the simple roots are uniquely determined up to rotation, reflection, and rescaling transformations. These transformations correspond to change of bases and rescaling of the Cartan subalgebra, which has no effect on the full representation.

3.2 Root Construction

We now follow the procedure outlined in Ref. [6] Sec. 8.2.6 and illustrated for G_2 in Sec. 8.6 for obtaining all the roots from the simple roots. We begin with whatever root we have so far, say ϕ (the base case is the simple roots) and attempt to raise using the simple roots. From the ‘master formula,’ we have

$$\frac{2\alpha^i \cdot \phi}{\alpha^{i2}} = -(p^i - q^i), \quad (3.7)$$

where p^i and q^i give respectively the number of times ϕ can be raised and lowered by simple root α^i without being annihilated. By the nature of this procedure, we always know the q^i values from the previous iterations (for the first iteration with simple roots, all $q^i = 0$ by the definition of simple roots, see Ref. [6] 8.2.1: $\alpha^i - \alpha^j$ is not a simple root).

Alternatively, this can be done graphically as described in Ref. [6] Sec. 8.8. Using the Cartan matrix, it can be shown that the $q^i - p^i$ values are additive. That is, when we raise the root $\phi \rightarrow \phi + \psi$ the new $q^i - p^i$ value is given (with the index i suppressed⁴)

$$q_\phi - p_\phi \rightarrow (q_\phi - p_\phi) + (q_\psi - p_\psi), \quad (3.8)$$

where q_ϕ indicates the q value of the root ϕ , etc. Hence, we may keep track of these values instead of the root vector itself. In this procedure, we record the $q^i - p^i$ value for each root, determine p^i knowing q^i from previous iterations, and write down the $q^i - p^i$ values of the appropriately raised roots by simply adding. The resulting root diagram is shown in Fig 3.2.

⁴The notation q with upper index suppressed can be interpreted as a vector indexed by i , that is $q = (q_1, q_2, \dots)$. This notation applies also to α (roots) and H (Cartan generators) with indices suppressed.

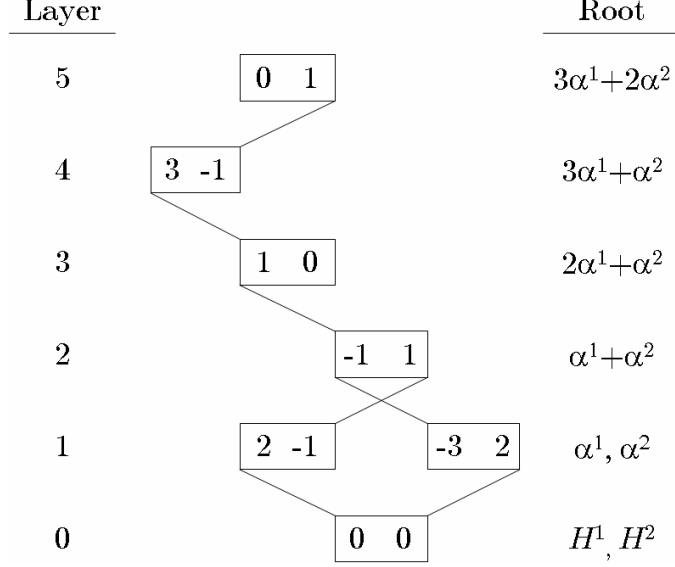


Figure 3.2: Complete positive root system of G_2 and graphical derivation. The layer column denotes how many simple roots the current root decomposes into; in boxes, the $q^i - p^i$ values are listed for each root; the root column gives the simple root decomposition.

3.3 Fundamental Weights

Ultimately, we are interested in constructing representations of the Lie algebra. A highest weight μ of some irreducible representation must have the property that $\mu + \phi$ is not a weight in the representation for any positive root ϕ , so that the weight cannot be raised any higher. Since all positive roots can be written as sums of simple roots, which are themselves positive, this is clearly equivalent to the property that $\mu + \alpha^i$ is not a weight for any simple root α^i . Therefore, for such a weight μ , we have $p^i = 0$ for all i , and so

$$\frac{2\alpha^i \cdot \mu}{\alpha^{i2}} = \ell^i, \quad (3.9)$$

where ℓ^i are nonnegative integers. Since the simple roots form a basis, μ is uniquely determined by the ℓ^i values, known as Dynkin coefficients. A convenient choice is to consider the set of fundamental weight vectors μ^i , which are defined by

$$\frac{2\alpha^i \cdot \mu^j}{\alpha^{i2}} = \delta_{ij}. \quad (3.10)$$

The representations D^i produced from these fundamental weights are known as fundamental representations. Then for a general highest weight μ , we have

$$\mu = \sum_i \ell^i \mu^i, \quad (3.11)$$

and the corresponding representation can be constructed by taking tensor products

$$\bigotimes_i \left[\underbrace{D^i \otimes D^i \otimes \cdots \otimes D^i}_{(\ell^i \text{ times})} \right]. \quad (3.12)$$

For an arbitrary Lie algebra, it is straightforward to obtain fundamental weights from the simple roots using Eq. (3.10). The direction of a particular fundamental weight, μ^i is fixed by the $N - 1$ orthogonality conditions

$$\alpha^j \cdot \mu^i = 0 \quad (3.13)$$

for $i \neq j$. The normalization of μ^i is determined by the remaining condition

$$\frac{2\alpha^i \cdot \mu^i}{\alpha^{i2}} = 1. \quad (3.14)$$

These conditions are nicely rewritten as the following matrix equation

$$\begin{pmatrix} \mu^1 \\ \mu^2 \\ \vdots \\ \mu^n \end{pmatrix} = \begin{pmatrix} \frac{2\alpha^1}{\alpha^{12}} & \frac{2\alpha^2}{\alpha^{22}} & \cdots & \frac{2\alpha^1}{\alpha^{n2}} \end{pmatrix}^{-1}. \quad (3.15)$$

For the G_2 algebra, we obtain two fundamental weights (recall that G_2 is rank two, and so there are two simple roots and two fundamental weights)

$$\mu^1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \right), \quad (3.16)$$

$$\mu^2 = (\sqrt{3}, 0). \quad (3.17)$$

3.4 Weight Construction

To obtain the complete weight system of a fundamental representation, we complete a similar analysis as described in Sec. 3.2 for constructing the complete root system. Again, we will record the $q^i - p^i$ values of the weights, which is given by the ‘master formula’, Eq. (3.7). This time, we are working down through the weights - we know all p^i values from previous iterations. For the base case, $p^i = 0$ for the fundamental weight, since it is the highest weight in the representation. As it turns out, in G_2 both of the fundamental weights are in fact also roots. Therefore, the process will produce a subset of the full root system obtained in Fig. 3.2. Furthermore, μ^2 is actually the highest root, so that in this fundamental representation we obtain the entire root system. This representation is in fact related to the adjoint representation by a similarity transformation. The full weight system for μ^1 is shown in Fig. 3.4, and for μ^2 in Fig. 3.4.

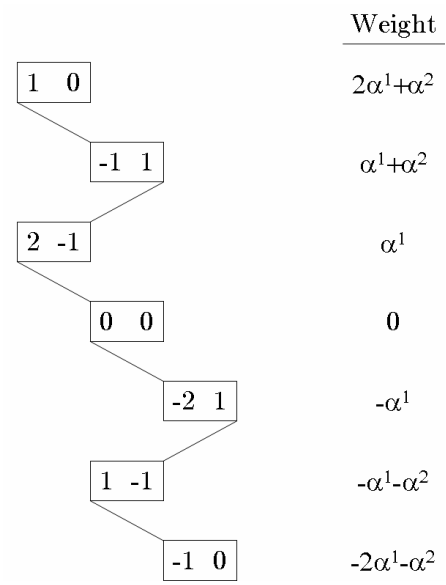


Figure 3.3: Weight diagram for fundamental representation with highest weight μ^1 .

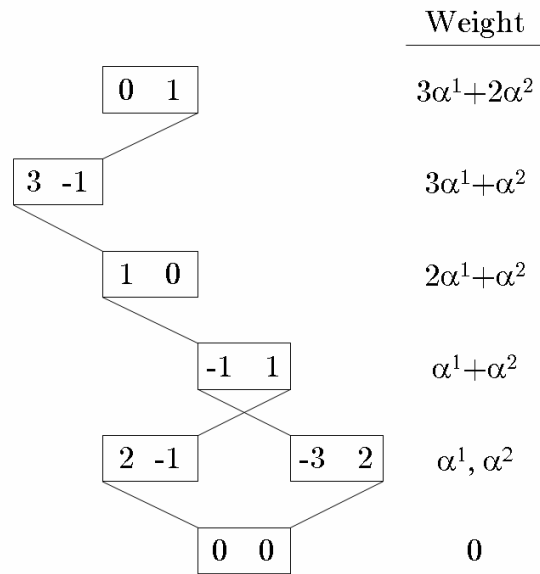


Figure 3.4: Weight diagram for fundamental representation with highest weight μ^2 . Only positive weights are shown; the full weight diagram is obtained by appending all negations of weights shown.

Chapter 4

Construction of the Fundamental Representation

In this chapter, we will use the root and weight constructions completed in the preceding chapter to produce representations of the Lie algebra and group. Here we are mainly interested on obtaining the 7-dimensional fundamental representation of G_2 ; from here on when we refer to the fundamental of G_2 , we mean this 7-dimensional representation. The adjoint representation is easily accessible once we have any other representation, since it is given by the action of the generators on each other.

4.1 Raising and Lowering Operators

From the root and weight analysis completed in Sec 3.3, we now wish to construct the fundamental representation of G_2 . Fig. 3.4 gives the 7-weight construction that corresponds to the 7-dimensional fundamental representation. For simplicity, we use the notation for the standard basis of the 7-dimensional space, where $|e_i\rangle$ is a vector whose j th entry is δ_{ij} . We take the weights in descending order, so that the highest weight $(2\alpha^1 + \alpha^2)$ corresponds to the vector $|e_1\rangle$, and the lowest weight $(-2\alpha^1 - \alpha^2)$ corresponds to the vector $|e_7\rangle$.

The Cartan generators are obtained by forming diagonal matrices from the first and second entries of the weight vectors, respectively, so as to satisfy the formula

$$H_i |\phi\rangle = \phi_i |\phi\rangle, \quad (4.1)$$

where ϕ is some weight vector and $|\phi\rangle$ is the state in the fundamental representation corre-

sponding to that weight. That is,

$$H_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix},$$

$$H_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

To construct the other generators of this representation, we will make use of the $SU(2)$ subalgebras associated with the roots from the construction in Fig 3.2. We should consider the action of the $SU(2)$'s on the states of the fundamental representation in the weight diagram, Fig 3.4.

First, recall the relationship between the $SU(2)$ raising and lowering operators E_α^\pm and the generators $E_{\pm\alpha}$ associated with some root α

$$E_{\pm\alpha} = |\alpha| E_\alpha^\pm. \quad (4.2)$$

The action of arbitrary $SU(2)$ raising and lowering operators is given as

$$J^+ |j, m\rangle = \sqrt{(j+m+1)(j-m)/2} |j, m+1\rangle, \quad (4.3)$$

$$J^- |j, m\rangle = \sqrt{(j+m)(j-m+1)/2} |j, m-1\rangle, \quad (4.4)$$

where $|j, m\rangle$ designates the spin state m in spin representation j .

We begin with the simple roots. For α^1 there are two doublets (e_1, e_2 and e_6, e_7) and one triplet (e_3, e_4, e_5) in the weight construction. This gives the following identification of states

in the $SU(2)$ s

$$\begin{aligned}
|e_1\rangle &= \left| \frac{1}{2}, \frac{1}{2}, 1 \right\rangle, \\
|e_2\rangle &= \left| \frac{1}{2}, -\frac{1}{2}, 1 \right\rangle, \\
|e_3\rangle &= |1, 1, 2\rangle, \\
|e_4\rangle &= |1, 0, 2\rangle, \\
|e_5\rangle &= |1, -1, 2\rangle, \\
|e_6\rangle &= \left| \frac{1}{2}, \frac{1}{2}, 3 \right\rangle, \\
|e_7\rangle &= \left| \frac{1}{2}, -\frac{1}{2}, 3 \right\rangle.
\end{aligned}$$

For this ket notation, we use a third index i to differentiate the three distinct representations of $SU(2)$ here, so we have kets of the form $|j, m, i\rangle$. Then we can compute the action of the raising operator on each state as follows

$$\begin{aligned}
E_{\alpha^1}^+ |e_1\rangle &= 0, \\
E_{\alpha^1}^+ |e_2\rangle &= \sqrt{1/2} |e_1\rangle, \\
E_{\alpha^1}^+ |e_3\rangle &= 0, \\
E_{\alpha^1}^+ |e_4\rangle &= |e_3\rangle, \\
E_{\alpha^1}^+ |e_5\rangle &= |e_4\rangle, \\
E_{\alpha^1}^+ |e_6\rangle &= 0, \\
E_{\alpha^1}^+ |e_7\rangle &= \sqrt{1/2} |e_6\rangle.
\end{aligned}$$

Finally, this yields the matrix form of the raising operator and the respective generator

$$E_{\alpha^1} = E_{\alpha^1}^+ = \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Taking the hermitian conjugate gives the generator associated with $-\alpha^1$

$$E_{-\alpha^1} = E_{\alpha^1}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{2}} & 0 \end{pmatrix}.$$

Repeating this procedure for α^2 , we obtain

$$E_{\alpha^2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_{-\alpha^2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \end{pmatrix}.$$

From here there are two ways to complete the construction of the representation, discussed in the next two sections.

4.2 Explicit Construction of the Representation

The first method is to continue as above, identifying the $SU(2)$ subalgebras associated with the non-simple roots and using them to explicitly write down the remaining generators associated with all the roots. A caveat for this approach: sign ambiguities arise and must be carefully considered. The difficulties, however, are purely notational.

As an example, consider the generator $E_{\alpha^1+\alpha^2}$. Referring to the weight diagram, we identify two doublets and a triplet. However, before we can construct the generator, we need to specify whether we are raising by α^1 or α^2 first. Although addition of vectors (and therefore of weights and roots) is commutative, raising and lowering operations are not. In

particular, we have the identity

$$E_{\alpha^2+\alpha^1} = -E_{\alpha^1+\alpha^2},$$

which arises from the commutation rule

$$[E_{\alpha^1}, E_{\alpha^2}] = \nu E_{\alpha^1+\alpha^2}.$$

where ν is a positive real parameter that can be determined from further considerations¹). Interchanging α^1 and α^2

$$[E_{\alpha^2}, E_{\alpha^1}] = \nu E_{\alpha^2+\alpha^1}.$$

As a result we get the strange notational result mentioned above: $E_{\alpha^2+\alpha^1} = -E_{\alpha^1+\alpha^2}$

The specification of ν is actually a conventional choice. In fact only the normalization of ν is fixed by the theory; its complex phase is undetermined. However, we chose the state $E_{\alpha^1+\alpha^2}$ such that ν is positive real. Notice that we are free to add any arbitrary phase to a state in the adjoint representation, and the state would maintain the same root vector. In particular we could add a phase of -1 , which produces the state we would refer to as $E_{\alpha^2+\alpha^1}$. Ultimately this is all meaningless for the algebra itself. The algebra is only a basis for a linear space, so we only need the one conventional phase, and all others such as $E_{\alpha^2+\alpha^1}$ are essentially irrelevant. The important result of the anticommutativity is that some entries of the generator obtain minus signs that would otherwise be missed.

If the raising operation in the weight construction occurs as written² in $\alpha_1 + \alpha_2$, that is, the weight is raised first by α^2 and then by α^1 , then the sign is positive as expected in the $SU(2)$ subalgebras. However, if the weight is raised in the opposite order in the weight construction, a minus sign is acquired since the raising actually corresponds to the $E_{\alpha^2+\alpha^1}$ operator. For $E_{\alpha^1+\alpha^2}$, there are two doublets (e_1, e_3 and e_5, e_7) and a triplet (e_2, e_4, e_6). The raising occurs as written for e_3 and e_6 , and in the opposite order for e_4 and e_7 . Therefore, we obtain

$$E_{\alpha^1+\alpha^2} = \begin{pmatrix} 0 & 0 & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

¹The exact relationship is obtained with the commutation rules that define the algebra, Eq. (4.9)

²Read addition right to left to mimic left multiplication

$$E_{-\alpha^2-\alpha^1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{1}{2}} & 0 & 0 \end{pmatrix}.$$

Notice that $E_{\alpha^1+\alpha^2}^\dagger = E_{-\alpha^2-\alpha^1}$. It is written this way because in lowering, the operations are done in reverse order, so that the weight is lowered first by α^1 and then by α^2 . The situation becomes more abstruse for longer sums of simple roots, where the sign is determined by whether the order of raising in the construction is an even (+) or odd (−) permutation of the order written. Completing this procedure we may compute the remaining generators.

4.3 Construction via Commutators of the Algebra

The second method is to compute the commutation relations for the algebra by using the raising and lowering operators for the simple roots and referring to the root construction. Again we consider $E_{\alpha^1+\alpha^2}$ as an example. Begin with the state in the adjoint representation corresponding to α^2

$$|E_{\alpha^2}\rangle = \left| \frac{3}{2}, -\frac{3}{2}, \alpha^1 \right\rangle$$

since it is the lowest state of a spin 3/2 representation (with respect to α^1). Recall the following for the adjoint representation

$$E_{\alpha^1} |E_{\alpha^2}\rangle = |[E_{\alpha^1}, E_{\alpha^2}]\rangle. \quad (4.5)$$

We know that $E_{\alpha^1} = E_{\alpha^1}^+$ from Eq. (4.5). Furthermore, for the action of raising and lowering operators in an arbitrary $SU(2)$ representation, we have

$$J^+ |j, m\rangle = \sqrt{(j+m+1)(j-m)/2} |j, m+1\rangle, \quad (4.6)$$

$$J^- |j, m\rangle = \sqrt{(j+m)(j-m+1)/2} |j, m-1\rangle. \quad (4.7)$$

So for the $E_{\alpha^1}^+$ raising operator, we have

$$E_{\alpha^1}^+ \left| \frac{3}{2}, -\frac{1}{2}, \alpha^1 \right\rangle = \sqrt{3/2} \left| \frac{3}{2}, -\frac{1}{2}, \alpha^1 \right\rangle.$$

Thus the following sequence of equalities holds

$$|[E_{\alpha^1}, E_{\alpha^2}]\rangle = E_{\alpha^1} |E_{\alpha^2}\rangle = E_{\alpha^1}^+ |E_{\alpha^2}\rangle = \sqrt{3/2} \left| \frac{3}{2}, -\frac{1}{2}, \alpha^1 \right\rangle = \sqrt{3/2} |E_{\alpha^1+\alpha^2}\rangle. \quad (4.8)$$

Or, in terms of the algebra

$$E_{\alpha^1+\alpha^2} = \sqrt{2/3} [E_{\alpha^1}, E_{\alpha^2}]. \quad (4.9)$$

We repeat this process for the other roots to obtain the commutations that define the algebra. For G_2 , these are as follows

$$E_{\alpha^1+\alpha^2} = \sqrt{2/3} [E_{\alpha^1}, E_{\alpha^2}], \quad (4.10)$$

$$E_{2\alpha^1+\alpha^2} = \sqrt{1/3} [E_{\alpha^1}, [E_{\alpha^1}, E_{\alpha^2}]], \quad (4.11)$$

$$E_{3\alpha^1+\alpha^2} = \sqrt{2/3} [E_{\alpha^1}, [E_{\alpha^1}, [E_{\alpha^1}, E_{\alpha^2}]]], \quad (4.12)$$

$$E_{\alpha^2+3\alpha^1+\alpha^2} = \frac{2}{3\sqrt{3}} [E_{\alpha^2}, [E_{\alpha^1}, [E_{\alpha^1}, [E_{\alpha^1}, E_{\alpha^2}]]]]. \quad (4.13)$$

Thereby we can write all the other generators in terms of the generators corresponding to the simple roots. The respective matrices can then be computed without explicitly constructing them as above. This method also makes it easier to locate the sign and notational ambiguities previously mentioned.

4.4 Complete Fundamental Representation

The representation can be completely constructed using either method above. The 14 resulting 7×7 matrices are listed below.

$$\begin{aligned}
 H_1 &= \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix} & H_2 &= \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \\
 E_{\alpha^1} &= \begin{pmatrix} 0 & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & E_{-\alpha^1} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{2}} & 0 \end{pmatrix} \\
 E_{\alpha^2} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & E_{-\alpha^2} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

$$E_{\alpha^1+\alpha^2} = \begin{pmatrix} 0 & 0 & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{-\alpha^2-\alpha^1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{1}{2}} & 0 & 0 \end{pmatrix}$$

$$E_{2\alpha^1+\alpha^2} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{-\alpha^2-2\alpha^1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{3\alpha^1+\alpha^2} = \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{-\alpha^2-3\alpha^1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{\alpha^2+3\alpha^1+\alpha^2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_{-\alpha^2-3\alpha^1-\alpha^2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Chapter 5

Useful Bases for Fundamental Representation

Recall that in obtaining the generators of the algebra, we are only interested in a basis for a linear space. Therefore, a variety of manipulations can be performed on this set to obtain one with perhaps more useful properties. The inner product and induced norm we use on this linear space is given as

$$\langle A|B\rangle = \text{Tr} (A^\dagger B) . \quad (5.1)$$

The generators are conventionally taken orthogonal to each other with norm $\frac{1}{\sqrt{2}}$. In the form above, they are already orthogonal, but have an incorrect normalization, $\sqrt{3}$. For applications we will renormalize the generators by multiplying by $\frac{1}{6}$. Thus we will have

$$\text{Tr} (T^{a\dagger} T^b) = \frac{1}{2} \delta_{ab} . \quad (5.2)$$

It is also useful to take a hermitian basis for the generators. This can easily be obtained by the standard replacement for matrices

$$\{A, A^\dagger\} \mapsto \left\{ \frac{A + A^\dagger}{\sqrt{2}}, \frac{A - A^\dagger}{\sqrt{2}i} \right\} . \quad (5.3)$$

This produces new matrices span the same linear space, with the same normalization, orthogonality, and the added property of hermiticity. Finally, we are free to take any similarity transformation on the set of generators, since the action of the algebra is persevered under similarity transformations.

$$\{A\} \mapsto \{U^{-1}AU\} . \quad (5.4)$$

where S is some fixed invertible matrix. Different similarity transformations allow different subalgebras and structures to be more easily manipulated, so we will make reference to several as necessary.

5.1 Standard Basis

The standard form of matrices for the 7-dimensional representation will be denoted T^a ; this basis is favored because it generalizes the form of the Gell-Mann matrices and makes the $SU(3)$ subgroup easily accessible. These are obtained from the E_α generators above by applying the renormalization and hermiticity operations described above, and then a similarity transform given by a unitary matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \quad (5.5)$$

The full formulation of T^a is thus obtained by¹

$$\{E_\alpha, E_{-\alpha}\} \mapsto \frac{1}{2\sqrt{3}} \{U^\dagger(E_\alpha + iE_{-\alpha})U, U^\dagger(iE_{-\alpha} - iE_\alpha)U\}. \quad (5.6)$$

The resulting matrices are given below, and written in simplified notation using the Gell-Mann matrices, λ^{1-8} .

$$T^1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & -\lambda_1^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & -\lambda_2^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

¹Actually, for some cases we take $\frac{1}{2\sqrt{3}} \{U^\dagger(-E_\alpha - E_{-\alpha})U, U^\dagger(iE_{-\alpha} - iE_\alpha)U\}$ to correct a sign flip that occurs due to one of the arbitrary choices of phase.

$$T^3 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & -\lambda_3^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^4 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \lambda_4 & 0 & 0 \\ 0 & -\lambda_4^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^5 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \lambda_5 & 0 & 0 \\ 0 & -\lambda_5^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^6 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \lambda_6 & 0 & 0 \\ 0 & -\lambda_6^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^7 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \lambda_7 & 0 & 0 \\ 0 & -\lambda_7^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^8 = \frac{1}{2\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \lambda_8 & 0 & 0 \\ 0 & -\lambda_8^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^9 = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & -i\lambda_7 & \sqrt{2}e_1 \\ i\lambda_7 & 0 & \sqrt{2}e_1 \\ \sqrt{2}e_1^T & \sqrt{2}e_1^T & 0 \end{pmatrix}$$

$$T^{10} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} \\ 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 \\ -i\sqrt{2} & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & -\lambda_7 & i\sqrt{2}e_1 \\ -\lambda_7 & 0 & -i\sqrt{2}e_1 \\ -i\sqrt{2}e_1^T & i\sqrt{2}e_1^T & 0 \end{pmatrix}$$

$$T^{11} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & i\lambda_5 & \sqrt{2}e_2 \\ -i\lambda_5 & 0 & \sqrt{2}e_2 \\ \sqrt{2}e_2^T & \sqrt{2}e_2^T & 0 \end{pmatrix}$$

$$T^{12} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2} \\ 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} \\ i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i\sqrt{2} & 0 & 0 & i\sqrt{2} & 0 & 0 \end{pmatrix} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & \lambda_5 & i\sqrt{2}e_2 \\ \lambda_5 & 0 & -i\sqrt{2}e_2 \\ -i\sqrt{2}e_2^T & i\sqrt{2}e_2^T & 0 \end{pmatrix}$$

$$\begin{aligned}
T^{13} &= \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{2} & 0 \end{pmatrix} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & -i\lambda_2 & \sqrt{2}e_3 \\ i\lambda_2 & 0 & \sqrt{2}e_3 \\ \sqrt{2}e_3^T & \sqrt{2}e_3^T & 0 \end{pmatrix} \\
T^{14} &= \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\sqrt{2} \\ 0 & i & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} \\ 0 & 0 & -i\sqrt{2} & 0 & 0 & i\sqrt{2} & 0 \end{pmatrix} = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & -\lambda_2 & i\sqrt{2}e_3 \\ -\lambda_2 & 0 & -i\sqrt{2}e_3 \\ -i\sqrt{2}e_3^T & i\sqrt{2}e_3^T & 0 \end{pmatrix}
\end{aligned}$$

In the equations above we use e_i to represent the standard basis vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

5.2 Eigenstate Basis

In obtaining the $SU(3)$ and $SU(2) \times U(1)$ decompositions of the adjoint representation of G_2 , one immediately notices that the standard basis is not given in terms of eigenstates of the Cartan generators of these subgroups ($T^{3,8}$). For example, for T^1 , we have

$$T^3|T^1\rangle = \frac{1}{4} \begin{pmatrix} i\lambda_2 & 0 & 0 \\ 0 & i\lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and for T^2 we have

$$T^3|T^2\rangle = \frac{1}{4} \begin{pmatrix} -i\lambda_1 & 0 & 0 \\ 0 & i\lambda_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, if we consider $T_1 \pm iT^2$, we find

$$T^3|T^1 \pm iT^2\rangle = \pm \frac{1}{2\sqrt{2}}(T^1 \pm iT^2).$$

Following this procedure, we obtain the complete eigenstate basis for G_2 , noting that the generators $T^{3,8}$ have a simultaneous eigenspace by the properties of Cartan generators. We

can label the new eigenstate basis as $\{\tilde{T}^a\}$, where we have

$$\begin{aligned}
\tilde{T}^1 &= \frac{1}{\sqrt{2}}(T^1 + iT^2) & \tilde{T}^8 &= T^8 \\
\tilde{T}^2 &= \frac{1}{\sqrt{2}}(T^1 - iT^2) & \tilde{T}^9 &= \frac{1}{\sqrt{2}}(T^9 - iT^{10}) \\
\tilde{T}^3 &= T^3 & \tilde{T}^{10} &= \frac{1}{\sqrt{2}}(T^{11} - iT^{12}) \\
\tilde{T}^4 &= \frac{1}{\sqrt{2}}(T^4 + iT^5) & \tilde{T}^{11} &= \frac{1}{\sqrt{2}}(T^{13} - iT^{14}) \\
\tilde{T}^5 &= \frac{1}{\sqrt{2}}(T^6 + iT^7) & \tilde{T}^{12} &= \frac{1}{\sqrt{2}}(T^9 + iT^{10}) \\
\tilde{T}^6 &= \frac{1}{\sqrt{2}}(T^4 - iT^5) & \tilde{T}^{13} &= \frac{1}{\sqrt{2}}(T^{11} + iT^{12}) \\
\tilde{T}^7 &= \frac{1}{\sqrt{2}}(T^6 - iT^7) & \tilde{T}^{14} &= \frac{1}{\sqrt{2}}(T^{13} + iT^{14})
\end{aligned}$$

For $a = 1 \dots 8$, we have

$$\tilde{T}^a = \frac{1}{2} \begin{pmatrix} K(a) & 0 & 0 \\ 0 & -K(a)^T & 0 \end{pmatrix},$$

where

$$\begin{aligned}
K(1) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & K(2) &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & K(3) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
K(4) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & K(5) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & K(7) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
K(7) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & K(8) &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned}$$

The remaining generators are given below,

$$\begin{aligned}
\tilde{T}^9 &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & \sqrt{2}e_1 \\ i\lambda_7 & 0 & 0 \\ 0 & \sqrt{2}e_1^T & 0 \end{pmatrix} & \tilde{T}^{12} &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & -i\lambda_7 & 0 \\ 0 & 0 & \sqrt{2}e_1 \\ \sqrt{2}e_1^T & 0 & 0 \end{pmatrix} \\
\tilde{T}^{10} &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & \sqrt{2}e_2 \\ -i\lambda_5 & 0 & 0 \\ 0 & \sqrt{2}e_2^T & 0 \end{pmatrix} & \tilde{T}^{13} &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & i\lambda_5 & 0 \\ 0 & 0 & \sqrt{2}e_2 \\ \sqrt{2}e_2^T & 0 & 0 \end{pmatrix} \\
\tilde{T}^{11} &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & \sqrt{2}e_3 \\ i\lambda_2 & 0 & 0 \\ 0 & \sqrt{2}e_3^T & 0 \end{pmatrix} & \tilde{T}^{14} &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & -i\lambda_2 & 0 \\ 0 & 0 & \sqrt{2}e_3 \\ \sqrt{2}e_3^T & 0 & 0 \end{pmatrix}
\end{aligned}$$

These six generators can be neatly summarized as follows; for $a = 9 \dots 11$,

$$\tilde{T}^a = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & \sqrt{2}\mu(a) \\ M(\mu(a)) & 0 & 0 \\ 0 & \sqrt{2}\mu(a)^T & 0 \end{pmatrix},$$

and for $a = 12 \dots 14$,

$$\tilde{T}^a = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & M(\mu(a))^T & 0 \\ 0 & 0 & \sqrt{2}\mu(a) \\ \sqrt{2}\mu(a)^T & 0 & 0 \end{pmatrix}.$$

Here, $\mu(a)$ are standard basis vectors: $\mu(9) = \mu(12) = e_1$, $\mu(10) = \mu(13) = e_2$, and $\mu(11) = \mu(14) = e_3$. The matrix M is given in terms of the Levi-Civita symbol, ϵ_{ijk} as

$$[M(\mu)]_{ij} = \epsilon_{ijk}\mu_k. \quad (5.7)$$

If we have $A = A^a T^a$ in the adjoint of G_2 , we may write it in the eigenstate basis as $A = \tilde{A}^a \tilde{T}^a$. Noting the identity $A^a T^a + A^b T^b = \left(\frac{A^a - iA^b}{\sqrt{2}}\right) \left(\frac{T^a + iT^b}{\sqrt{2}}\right) + \left(\frac{A^a + iA^b}{\sqrt{2}}\right) \left(\frac{T^a - iT^b}{\sqrt{2}}\right)$, we can obtain

$$\begin{aligned} \tilde{A}^1 &= \frac{1}{\sqrt{2}}(A^1 - iA^2) & \tilde{A}^8 &= A^8 \\ \tilde{A}^2 &= \frac{1}{\sqrt{2}}(A^1 + iA^2) & \tilde{A}^9 &= \frac{1}{\sqrt{2}}(A^9 + iA^{10}) \\ \tilde{A}^3 &= A^3 & \tilde{A}^{10} &= \frac{1}{\sqrt{2}}(A^{11} + iA^{12}) \\ \tilde{A}^4 &= \frac{1}{\sqrt{2}}(A^4 - iA^5) & \tilde{A}^{11} &= \frac{1}{\sqrt{2}}(A^{13} + iA^{14}) \\ \tilde{A}^5 &= \frac{1}{\sqrt{2}}(A^4 + iA^5) & \tilde{A}^{12} &= \frac{1}{\sqrt{2}}(A^9 - iA^{10}) \\ \tilde{A}^6 &= \frac{1}{\sqrt{2}}(A^6 - iA^7) & \tilde{A}^{13} &= \frac{1}{\sqrt{2}}(A^{11} - iA^{12}) \\ \tilde{A}^7 &= \frac{1}{\sqrt{2}}(A^6 + iA^7) & \tilde{A}^{14} &= \frac{1}{\sqrt{2}}(A^{13} - iA^{14}) \end{aligned}$$

The decompositions we will obtain for $SU(3)$ and $SU(2) \times U(1)$ subgroups can be written more compactly in terms of the \tilde{A}^a . Furthermore, the \tilde{A}^a terms will correspond to the physical gauge bosons and mass eigenstates when we build models with G_2

5.3 $SU(3)$ Subgroup

The $SU(3)$ subgroup is formed by T^{1-8} . This basis also makes the decomposition of the fundamental representation of G_2 under $SU(3)$ clear:

$$\mathbf{7} = \begin{pmatrix} \mathbf{3} \\ \mathbf{\bar{3}} \\ \mathbf{1} \end{pmatrix}. \quad (5.8)$$

The generators T^{9-14} can be rewritten as

$$T^a = \frac{1}{2\sqrt{6}} \begin{pmatrix} 0 & M(\chi(a))^\dagger & \sqrt{2}\chi(a) \\ M(\chi(a)) & 0 & \sqrt{2}\chi(a)^* \\ \sqrt{2}\chi(a)^\dagger & \sqrt{2}\chi(a)^T & 0 \end{pmatrix},$$

where $\chi(9) = e_1, \chi(10) = ie_1, \chi(11) = e_2, \chi(12) = ie_2, \chi(13) = e_3, \chi(14) = ie_3$, and $M(\chi)$ is given as in Eq. (5.7). This form indicates that the decomposition of the 14-dimensional adjoint representation of G_2 under $SU(3)$ as

$$\mathbf{14} = \mathbf{8} + \mathbf{3} + \mathbf{\bar{3}} \quad (5.9)$$

where the $\mathbf{8}$ is formed by T^{1-8} , and the $\mathbf{3}, \bar{\mathbf{3}}$ correspond to the three complex degrees of freedom of χ . Note also that all the generators can be written in some neat way as a bordered matrix of some tensor product of Pauli or identity matrices with Gell-Mann matrices,

$$T^a = \begin{pmatrix} \sigma_i \otimes \lambda_j & \chi \\ \chi^\dagger & \chi^T \\ & & 0 \end{pmatrix},$$

where χ is some, possibly zero, 3-vector.

We can also check decomposition of Eq. (5.9) by computing the action of the generators on the adjoint representation, given by

$$T^a |T^b\rangle = |[T^a, T^b]\rangle, \quad (5.10)$$

and comparing to the action of known representations of $SU(3)$. It is clear that T^{1-8} forms an $\mathbf{8}$ under $SU(3)$. Suppose $A = A^a T^a$. We find by computing the action of the adjoint that

$$\frac{1}{\sqrt{2}} \begin{pmatrix} A^9 + iA^{10} \\ A^{11} + iA^{12} \\ A^{13} + iA^{14} \end{pmatrix} \sim \mathbf{3}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} A^9 - iA^{10} \\ A^{11} - iA^{12} \\ A^{13} - iA^{14} \end{pmatrix} \sim \bar{\mathbf{3}}. \quad (5.11)$$

5.4 $SU(2) \times U(1)$ Subgroup

All information about the $SU(2) \times U(1)$ subgroup of G_2 can be easily obtained from the $SU(3)$ form of the generators. This convenient situation follows because of the following inclusions: $SU(2) \times U(1) \subset SU(3) \subset G_2$. Let $t^a = \frac{1}{2}\lambda_a$ be the normalized generators of $SU(3)$. Because of our choice of basis, any conclusion about the $SU(2) \times U(1)$ in $SU(3)$ can be carried over to G_2 by maintaining the same indices and replacing ts by Ts ; we can therefore work with the simpler $SU(3)$ group and obtain many useful relations.

The $SU(2)$ subgroup of $SU(3)$ is formed by t^{1-3} , and easily recognized because it displays the Pauli matrices in the upper 2×2 block; the $U(1)$ subgroup is generated by the remaining Cartan generator, t^8 . Therefore, the $SU(2)$ and $U(1)$ subgroups of G_2 are generated respectively by T^{1-3} and T^8 .

Again, we can obtain the decomposition of any representation of $SU(3)$ under $SU(2) \times U(1)$ by considering the action of the subgroup on the representation. Assuming that the $U(1)$ generator is normalized by $Y = \sqrt{3}T^8$, we find that the decomposition of the fundamental is given by

$$\mathbf{3} = \begin{pmatrix} \mathbf{2}_{1/2} \\ \mathbf{1}_0 \end{pmatrix}, \quad (5.12)$$

and the adjoint by

$$\mathbf{8} = \mathbf{3}_0 + \mathbf{1}_0 + \mathbf{2}_{3/2} + \mathbf{2}_{-3/2}, \quad (5.13)$$

where, if we suppose $\Lambda = \Lambda^a t^a$, we have

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \Lambda^1 - i\Lambda^2 \\ \Lambda^1 + i\Lambda^2 \\ \sqrt{2}\Lambda^3 \end{pmatrix} \sim \mathbf{3}_0, \quad \Lambda^8 \sim \mathbf{1}_0, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} \Lambda^4 \mp i\Lambda^5 \\ \Lambda^6 \mp i\Lambda^7 \end{pmatrix} \sim \mathbf{2}_{\pm 3/2}$$

Comparing with Eqs. (5.9, 5.10), we obtain the following decompositions of G_2 under $SU(2) \times U(1)$. For the fundamental, we have²

$$\mathbf{7} = \begin{pmatrix} \mathbf{2}_{1/2} \\ \mathbf{1}_{-1} \\ \bar{\mathbf{2}}_{-1/2} \\ \mathbf{1}_1 \\ \mathbf{1}_0 \end{pmatrix}, \quad (5.14)$$

and for the adjoint, we have

$$\mathbf{14} = (\mathbf{3}_0 + \mathbf{1}_0 + \mathbf{2}_{3/2} + \mathbf{2}_{-3/2}) + (\mathbf{2}_{1/2} + \mathbf{1}_{-1}) + (\mathbf{2}_{-1/2} + \mathbf{1}_1), \quad (5.15)$$

with

$$\begin{aligned} \frac{1}{\sqrt{2}} \begin{pmatrix} A^1 - iA^2 \\ A^1 + iA^2 \\ \sqrt{2}A^3 \end{pmatrix} &\sim \mathbf{3}_0, \quad A^8 \sim \mathbf{1}_0, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} A^4 \mp iA^5 \\ A^6 \mp iA^7 \end{pmatrix} \sim \mathbf{2}_{\pm 3/2}, \\ \frac{1}{\sqrt{2}} \begin{pmatrix} A^9 \pm iA^{10} \\ A^{11} \pm iA^{12} \end{pmatrix} &\sim \mathbf{2}_{\pm 1/2}, \quad \frac{1}{\sqrt{2}}(A^{13} \pm iA^{14}) \sim \mathbf{1}_{\mp 1}. \end{aligned}$$

One can easily check that these decompositions agree with the ones given in Sec. 5.3.

5.5 $SU(2) \times SU(2)$ Form

An important alternative basis for the algebra presents the $SU(2) \times SU(2)$ subgroup in a useful way. We will denote the generators in this basis as \hat{T}^a . Consider the unitary matrix

$$\hat{U} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad (5.16)$$

If, instead of Eq. (5.6), we take the following transformation

$$\{E_\alpha, E_{-\alpha}\} \mapsto \frac{1}{2\sqrt{3}} \left\{ \hat{U}^\dagger (E_\alpha + E_{-\alpha}) \hat{U}, \hat{U}^\dagger (iE_{-\alpha} - iE_\alpha) \hat{U} \right\} \quad (5.17)$$

²Note that $\bar{\mathbf{2}}$ is isomorphic to $\mathbf{2}$, so we may drop the conjugation.

we obtain the following generators (up to reordering and sign changes).

$$\hat{T}^1 = \frac{1}{12} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} \sigma_1 \otimes \mathbf{1} & 0 \\ 0 & 2J_1 \end{pmatrix}$$

$$\hat{T}^2 = \frac{1}{12} \begin{pmatrix} 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} \sigma_2 \otimes \mathbf{1} & 0 \\ 0 & 2J_2 \end{pmatrix}$$

$$\hat{T}^3 = \frac{1}{12} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} \sigma_3 \otimes \mathbf{1} & 0 \\ 0 & 2J_3 \end{pmatrix}$$

$$\hat{T}^4 = \frac{1}{4\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4\sqrt{3}} \begin{pmatrix} \mathbf{1} \otimes \sigma_1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{T}^5 = \frac{1}{4\sqrt{3}} \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4\sqrt{3}} \begin{pmatrix} \mathbf{1} \otimes \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{T}^6 = \frac{1}{4\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4\sqrt{3}} \begin{pmatrix} \mathbf{1} \otimes \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{T}^7 = \frac{1}{4\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{T}^8 = \frac{1}{4\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{T}^9 = \frac{1}{4\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{T}^{10} = \frac{1}{4\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
\hat{T}^{11} &= \frac{1}{12} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\hat{T}^{12} &= \frac{1}{12} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 \\ -i\sqrt{2} & 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 \end{pmatrix} \\
\hat{T}^{13} &= \frac{1}{12} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \\
\hat{T}^{14} &= \frac{1}{12} \begin{pmatrix} 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\sqrt{2} & -i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Here σ_i are the Pauli matrices, and J_i are the spin 1 matrices, from the 3-dimensional representation of $SU(2)$:

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This basis gives the following decomposition of the fundamental representation of G_2 under $SU(2) \times SU(2)$

$$(\mathbf{7}) = \left(\begin{smallmatrix} \mathbf{2}, \mathbf{2} \\ \mathbf{3}, \mathbf{1} \end{smallmatrix} \right). \quad (5.18)$$

We can also use this basis to determine the decomposition of the 14-dimensional adjoint representation under $SU(2) \times SU(2)$, again by noting Eq. (5.10). Consequently, we find

$$\mathbf{14} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{4}, \mathbf{2}), \quad (5.19)$$

where, if we write $\hat{A} = \hat{A}^a \hat{T}^a$, we have

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \hat{A}^1 - i\hat{A}^2 \\ \hat{A}^1 + i\hat{A}^2 \\ \sqrt{2}\hat{A}^3 \end{pmatrix} \sim (\mathbf{3}, \mathbf{1}), \quad \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{A}^4 - i\hat{A}^5 \\ \hat{A}^4 + i\hat{A}^5 \\ \sqrt{2}\hat{A}^6 \end{pmatrix} \sim (\mathbf{1}, \mathbf{3}), \quad (5.20)$$

and

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \hat{A}^7 - i\hat{A}^8 \\ \hat{A}^9 - i\hat{A}^{10} \\ \hat{A}^{11} - i\hat{A}^{12} \\ \hat{A}^{13} - i\hat{A}^{14} \\ \hat{A}^{13} + i\hat{A}^{14} \\ \hat{A}^{11} + i\hat{A}^{12} \\ \hat{A}^9 + i\hat{A}^{10} \\ \hat{A}^7 + i\hat{A}^8 \end{pmatrix} \sim (\mathbf{4}, \mathbf{2}). \quad (5.21)$$

Chapter 6

$SU(3) \times SU(2) \times U(1)$ Electroweak Model

Here we will describe in detail the $SU(3) \times SU(2) \times U(1)$ electroweak model that was proposed in [3] and outlined in Sec. 2.3.

We denote the gauge couplings for $SU(3)$, $SU(2)$, $U(1)$ as $g_3, \tilde{g}, \tilde{g}'$ respectively. The $SU(3)$ group contains subgroups which we will denote $SU(2)_0, U(1)_0$. The full $SU(3) \times SU(2) \times U(1)$ gauge group will be broken to its diagonal subgroup, which will be identified with the Standard Model electroweak gauge group $SU(2)_W \times U(1)_Y$. As previously described, the Standard Model fields are included here in their usual representations, transforming only under the additional $SU(2) \times U(1)$.

6.1 Symmetry Breaking

The $SU(3) \times SU(2)_W \times U(1)_Y$ gauge group is broken to $SU(2) \times U(1)$ by adding a Higgs field $\Sigma \sim (\mathbf{3}, \bar{\mathbf{2}}_{-1/2})$ with vacuum expectation value

$$\langle \Sigma \rangle = \begin{pmatrix} M & 0 \\ 0 & M \\ 0 & 0 \end{pmatrix}. \quad (6.1)$$

This form for the Higgs field can be deduced by considering the appropriate Higgs for breaking $SU(3) \times SU(3)$ to its diagonal subgroup. An arbitrary element in the adjoint of $SU(3)$ is given by $\alpha^a t^a$; an element in the adjoint of $SU(2)$ is given by $\beta^a \tilde{t}^a$, and an element in the adjoint of $U(1)$ is given by δ . The expectation of the Higgs should be invariant under action of the gauge group; that is, for

$$\begin{aligned} \Sigma &\rightarrow e^{i\alpha^a t^a} \Sigma e^{-i\beta^b \tilde{t}^b} e^{-i\delta/2} \\ \langle \Sigma \rangle &\rightarrow \langle \Sigma \rangle \end{aligned}$$

Notice that the action of $U(1)$ is in general given by $e^{i\delta Y}$ and here $Y = -1/2$. Computing the expectation of the transformed Σ field, we find

$$\langle e^{i\alpha^a t^a} \Sigma e^{-i\beta^b \tilde{t}^b} e^{-i\delta/2} \rangle = e^{i\alpha^a t^a} \langle \Sigma \rangle e^{-i\beta^b \tilde{t}^b} e^{-i\delta/2}.$$

Then, due to the uniqueness of inverses in the group, we can conclude that

$$\alpha^a = \begin{cases} \beta^a & \text{for } a = 1, \dots, 3 \\ \sqrt{3}\delta & \text{for } a = 8 \\ 0 & \text{otherwise} \end{cases},$$

which is precisely the diagonal subgroup.

6.2 Particle Spectrum

The Higgs field Σ transforms in the full $SU(3) \times SU(2) \times U(1)$ as $(\mathbf{3}, \bar{\mathbf{2}}_{-1/2})$. Note that the $\mathbf{2}$ representation of $SU(2)$ is a real representation, so that $\mathbf{2}$ is isomorphic to $\bar{\mathbf{2}}$. This statement will be described in more detail in Ch. 7; for now we simply note that tensor products of representations can be computed replacing the $\bar{\mathbf{2}}$ by $\mathbf{2}$. The fundamental of $SU(3)$ decomposes under the $SU(2)_0 \times U(1)_0$ subgroups as

$$\mathbf{3} = \mathbf{2}_{1/2} + \mathbf{1}_{-1},$$

and so we can conclude that the field Σ decomposes under $SU(2)_W \times U(1)_Y$ as

$$\Sigma \sim (\mathbf{2}_{1/2} + \mathbf{1}_{-1}) \otimes \bar{\mathbf{2}}_{-1/2} = \mathbf{3}_0 + \mathbf{1}_0 + \mathbf{2}_{-3/2}.$$

It is assumed that these scalar fields obtain masses at the scale of the symmetry-breaking, M .

The $SU(3)$ gauge bosons $A = A^a t^a$ transform in the adjoint representation, which decomposes under $SU(2) \times U(1)$ as

$$\mathbf{8} = \mathbf{3}_0 + \mathbf{1}_0 + \mathbf{2}_{3/2} + \mathbf{2}_{-3/2}. \quad (6.2)$$

The $\mathbf{2}_{3/2}$ and $\mathbf{2}_{-3/2}$ representations form a complex doublet. When we include the additional $SU(2) \times U(1)$ factors, we find gauge bosons in the model with the following $SU(2)_W \times U(1)_Y$ representations:

$$\mathbf{3}_0 + \mathbf{1}_0 + \mathbf{3}_0 + \mathbf{1}_0 + \mathbf{2}_{\pm 3/2}. \quad (6.3)$$

The Lagrangian contains a kinetic term from the Σ field given by

$$\mathcal{L} = \text{Tr} \left[\left(ig_3 A^a t^a \Sigma - i\tilde{g} \tilde{W}^b \tilde{t}^b \Sigma - i\tilde{g}' \frac{B}{2} \Sigma \right)^\dagger \left(ig_3 A^a t^a \Sigma - i\tilde{g} \tilde{W}^b \tilde{t}^b \Sigma - i\tilde{g}' \frac{B}{2} \Sigma \right) \right].$$

Now we substitute the vacuum expectation value of Σ into this term. If we let $A^a = \tilde{W}^{a-8}$ for $a = 9 \dots 12$, $A^{13} = B$, and \mathbf{A} denote a vector with entries A^a , then this term can be rewritten as

$$\mathcal{L} = \frac{1}{2} \sum_{a,b} A^a (M_{sq})_{ab} A^b = \frac{1}{2} \mathbf{A}^T M_{sq} \mathbf{A},$$

where M_{sq} is a matrix whose entries are appropriately defined from the original kinetic term. For example, for $a, b = 1 \dots 8$, the (a, b) entry is given by $g_3^2 \Sigma^T (t^a t^b + t^b t^a) \Sigma$. We identify these terms with terms that would arise in the potential due to massive fields. By diagonalizing the matrix M_{sq} we can compute mass eigenstates which correspond to the physical gauge bosons.

The mass eigenstate $\mathbf{3}_0$ fields arise from mixing of the SU(2) gauge fields \widetilde{W}^a and the SU(2)₀ gauge fields A^a , for $a = 1, 2, 3$. In the (A^a, \widetilde{W}^a) basis, the mass squared matrix is

$$M^2 \begin{pmatrix} g_3^2 & -g_3 \tilde{g} \\ -g_3 \tilde{g} & \tilde{g}^2 \end{pmatrix}. \quad (6.4)$$

Therefore, one obtains the mass eigenstates

$$W_L^a = c_\phi A^a - s_\phi \widetilde{W}^a \quad (6.5)$$

$$W_H^a = s_\phi A^a + c_\phi \widetilde{W}^a \quad (6.6)$$

with

$$s_\phi = \frac{-g_3}{\sqrt{g_3^2 + \tilde{g}^2}} \quad \text{and} \quad c_\phi = \frac{\tilde{g}}{\sqrt{g_3^2 + \tilde{g}^2}}, \quad (6.7)$$

and the masses

$$M_{W_L} = 0 \quad (6.8)$$

$$M_{W_H} = (g_3^2 + \tilde{g}^2)^{1/2} M. \quad (6.9)$$

The mass eigenstate $\mathbf{1}_0$ fields arise from the mixing of the U(1) field \widetilde{B} and the U(1)₀ field A^8 . In the (A^8, \widetilde{B}) basis, the mass squared matrix is

$$(1 + 2x^2) M^2 \begin{pmatrix} g_3^2/3 & -g_3 \tilde{g}'/\sqrt{3} \\ -g_3 \tilde{g}'/\sqrt{3} & \tilde{g}'^2 \end{pmatrix}. \quad (6.10)$$

One immediately obtains the mass eigenstates

$$B_L = c_\psi A^8 - s_\psi \widetilde{B} \quad (6.11)$$

$$B_H = s_\psi A^8 + c_\psi \widetilde{B}, \quad (6.12)$$

with

$$s_\psi = \frac{-g_3}{\sqrt{g_3^2 + 3\tilde{g}'^2}}, \quad \text{and} \quad c_\psi = \frac{\sqrt{3}\tilde{g}'}{\sqrt{g_3^2 + 3\tilde{g}'^2}}, \quad (6.13)$$

and the masses

$$M_{B_L} = 0 \quad (6.14)$$

$$M_{B_H} = \left(\frac{g_3^2}{3} + \tilde{g}'^2 \right)^{1/2} M. \quad (6.15)$$

Finally, the $\mathbf{2}_{\pm 3/2}$ state is formed from the remaining components of the SU(3) adjoint, A^a for $a = 4, 5, 6, 7$. Its mass is given by

$$M_{3/2} = \frac{1}{\sqrt{2}} g_3 M, \quad (6.16)$$

where the subscript indicates the hypercharge of the state. The mass eigenstates can be obtained from analysis of $SU(3)$ or from our analysis of the $SU(3)$ subgroup of G_2 in Sec. 5.3. We find

$$\frac{1}{\sqrt{2}} \begin{pmatrix} A^4 \mp iA^5 \\ A^6 \mp iA^7 \end{pmatrix} \sim \mathbf{2}_{\pm 3/2}$$

For $g_3 \ll \tilde{g}, \tilde{g}'$, the $\mathbf{2}_{\pm 3/2}$ gauge bosons will be significantly lighter than the other massive bosons, W_H^a and B_H , and are therefore the best candidates for observation at particle colliders.

6.3 Standard Model Couplings

Now we are prepared to compute the Standard Model couplings g, g' . We write

$$g_3 A^a t^a = g W_L^a \tilde{t}^a + g' B_L Y + (\text{massive fields}).$$

Then we can immediately compute the couplings by inverting the relations for W_L^a and B_L above, and recalling that $Y = \sqrt{3}t^8$. We find

$$\begin{aligned} g &= g_3 c_\phi, \\ g' &= g_3 c_\psi / \sqrt{3}, \end{aligned}$$

or equivalently,

$$\frac{1}{g^2} = \frac{1}{g_3^2} + \frac{1}{\tilde{g}^2}, \tag{6.17}$$

$$\frac{1}{g'^2} = \frac{3}{g_3^2} + \frac{1}{\tilde{g}'^2}. \tag{6.18}$$

This result indicates that the $\sin^2 \theta_W = \frac{1}{4}$ prediction from the $SU(3)$ model carries over in the limit that $\tilde{g}, \tilde{g}' \gg g_3$. Therefore we can work in this limit and maintain the successful weak mixing angle prediction.

6.4 Precision Electroweak Analysis

The model predicts shifts to electroweak observables, which were computed and compared to precision electroweak data in [4]. The corrections to electroweak observables are given most conveniently in terms of two new parameters

$$c_1 = \frac{c_\phi^4 v^2}{4M^2}, \quad c_2 = \frac{c_\psi^4 v^2}{4M^2};$$

where $v = 246$ GeV is associated with the vacuum expectation value of the ordinary Higgs field, Φ ,

$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}.$$

The authors of [4] go on to compute the shifts to 22 precision electroweak observables that occur due to the new theory. These shifts are recorded in the appendix to this chapter. Using experimental values and standard errors, they perform a chi-squared fit on the parameters c_1, c_2 , and obtain confidence contours in $c_1 - c_2$ space. The experimental input parameters they use are given in Table 6.1.

The chi-squared analysis is as follows. Each observable is assumed to have a normal distribution with mean and standard deviation obtained from the experimental values and the standard errors; so each observable is distributed as $O_i \sim N(\mu_i, \sigma_i^2)$. The value of the observable from the new theory is computed in terms of c_1, c_2 as $T_i(c_1, c_2)$. Thus there is a chi-squared random variable given by

$$X(c_1, c_2) = \sum_{i=1}^{22} \left(\frac{T_i(c_1, c_2) - \mu_i}{\sigma_i} \right)^2; X \sim \chi^2(22),$$

with 22 degrees of freedom. The best fit (minimum) value of X over c_1, c_2 is given by $X_{min} = 30.5$. The new theory is taken to be the null hypothesis; then

$$X - X_{min} \sim \chi^2(2)$$

also has a chi-squared distribution with two degrees of freedom, for the remaining parameters, c_1, c_2 . Confidence contours are computed by setting $X(c_1, c_2) - X_{min} = q$, where q is the appropriate quantile of the $\chi^2(2)$ distribution, and then solving for c_2 in terms of c_1 . Since only first-order linear corrections were used, the resulting equation is quadratic polynomial that is easily solved. A plot containing the 68%, 95%, and 99% confidence contours in $c_1 - c_2$ space is given in Fig. 6.4.

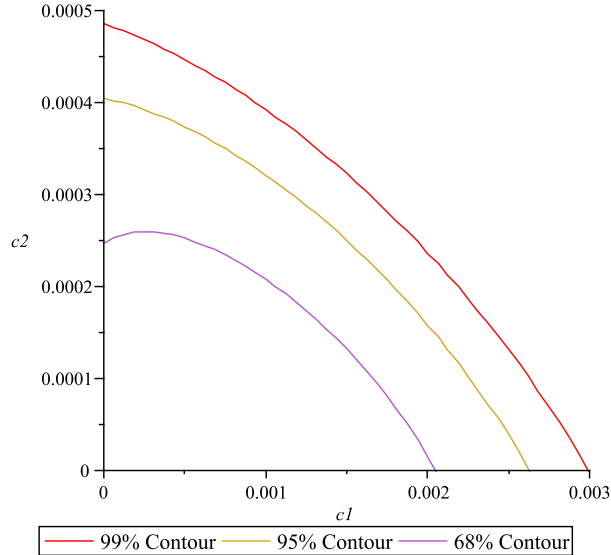


Figure 6.1: Confidence level contours in the $c_1 - c_2$ plane for the electroweak fit. The 95% contour is used to define physically excluded regions of parameter space.

The authors use a 95% confidence level to define physically excluded and allowable regions in parameter space. These parameters are then related back to the free parameters of the model, M, \tilde{g}, \tilde{g}' through the renormalization group equations. The renormalization equations also allow computation of couplings at the unification scale M_U from the known couplings at the scale of the Z boson, M_Z . The unification scale is taken to be the mass of the heaviest gauge boson, $M_U = \max\{M_{W_H}, M_{B_H}\}$. Denote $\alpha^{-1} = \frac{4\pi}{g^2}$ and $\alpha'^{-1} = \frac{4\pi}{g'^2}$. Then the renormalization group equations are given as

$$\alpha^{-1}(M_U) = \alpha^{-1}(M_Z) + \frac{b_{SM}}{2\pi} \ln \frac{M_U}{M_Z} + \sum_i \frac{b_i}{2\pi} \ln \frac{M_U}{M_i}, \quad (6.19)$$

$$\alpha'^{-1}(M_U) = \alpha'^{-1}(M_Z) + \frac{b'_{SM}}{2\pi} \ln \frac{M_U}{M_Z} + \sum_i \frac{b'_i}{2\pi} \ln \frac{M_U}{M_i}, \quad (6.20)$$

where M_i is the mass of the i th heavy particle scale, and b_i is the corresponding contribution to the beta functions. For this analysis, there is only one heavy scale, given by $M_{3/2} = g_3 M / \sqrt{2}$.

The computation of beta functions follows from the equation

$$\frac{d\alpha^{-1}}{dt} = \frac{1}{2\pi} \left(\frac{11}{3} S_1(G) - \frac{2}{3} S_3(F_L) - \frac{2}{3} S_3(F_R) - \frac{1}{3} S_3(Sc) \right),$$

where G is the gauge group associated with the coupling, $F_{L,R}$ are the representations of left- and right-handed fermions respectively, and Sc indicates the complex scalar representations; and

$$S_1(G) \delta_{ab} = f_{acd} f_{bcd}, \quad (6.21)$$

$$S_3(R) \delta_{ab} = \text{Tr}(T^a T^b). \quad (6.22)$$

Some useful values of S_1, S_3 are given below:

$$S_1(SU(N)) = N, \quad S_1(U(1)) = 0, \\ S_3(D_Q(U(1))) = Q^2, \quad S_3(D_F(G)) = \frac{1}{2}, \quad S_3(D_A(G)) = S_1(G),$$

where G is some arbitrary group, D_F, D_A are the fundamental and adjoint representations, respectively, and D_Q is the Q -charge representation of $U(1)$. The contributions to beta functions obtained from these formulae are $b_{SM} = 19/6$, $b'_{SM} = -41/6$, $b_{3/2} = 7/2$, $b'_{3/2} = 63/2$.

Every point in $\tilde{g} - \tilde{g}'$ parameter space can now be used to specify the complete theory. Eq. (6.17), which gives Standard Model couplings in terms of the high energy couplings can be used to write g, g' in terms of g_3 . From the computation of gauge boson masses, M_U, M_i can be eliminated in favor of M . Once a point in $\tilde{g} - \tilde{g}'$ parameter space is specified, this leaves two equations, Eq. (6.19), in two unknowns, g_3, M . Then the parameters c_1, c_2 are also completely specified, and can be compared to the confidence contours to generate excluded and allowable regions. Alternatively, one can take the equation for the $c_1 - c_2$ confidence contour as an additional equation in the system and solve for the $\tilde{g} - \tilde{g}'$ confidence contour, thereby generating excluded/allowable regions in $\tilde{g} - \tilde{g}'$ space.

The results of this analysis were presented in [4]. Here we include plots of $\tilde{g} - \tilde{g}'$ space that indicate excluded and allowable regions along with contours of unification scale M_U and intermediate mass scale M_i . The main result of this analysis was that for $\tilde{g}, \tilde{g}' > 1$, all values of M_U less than 11 TeV were excluded at the 95% confidence level. Values of M_i less than 2.5 TeV were excluded similarly. The unification scale is unambiguously outside of the range of near-term colliders, particularly the Large Hadron Collider at CERN. However, the $\mathbf{2}_{\pm 3/2}$ bosons have masses at the scale M_i , and are potentially just within reach of the LHC.

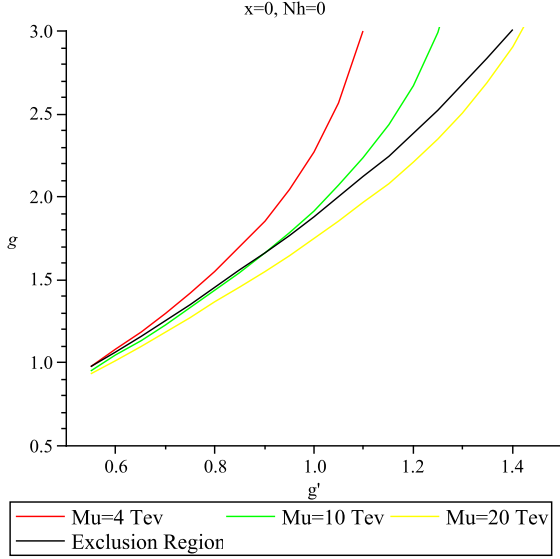


Figure 6.2: Contours of the unification scale M_U . The region above the black line is excluded at the 95% confidence level.

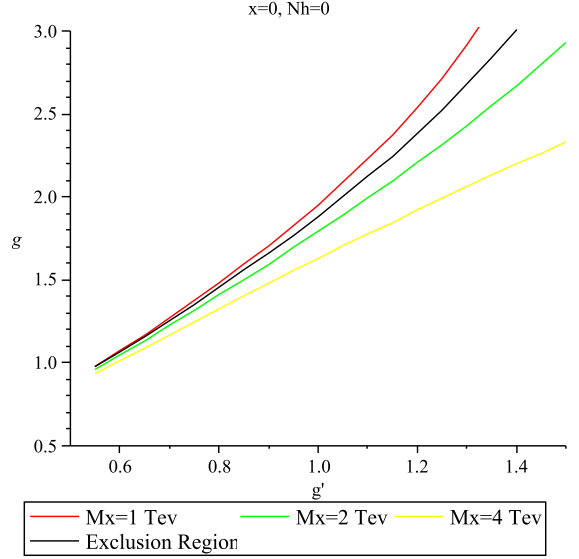


Figure 6.3: Contours of the mass of the lightest gauge boson, $\mathbf{2}_{\pm 3/2}$. The region above the black line is excluded at the 95% confidence level.

The authors go on to discuss the cosmological problem of the stable, charged $\mathbf{2}_{\pm 3/2}$ exotic gauge bosons, which do not couple to ordinary matter fields. They mention that the problem could be resolved by introducing additional fields that transform under the $SU(3)$.

Table 6.1: Input parameters for the electroweak fit described in the text. The SM column shows predictions in which $m_h = 115$ GeV and $\alpha_s = 0.120$.

Quantity	Experiment	SM	Quantity	Experiment	SM
Γ_Z	2.4952 ± 0.0023	2.4965	$A_e(P_\tau)$	0.15138 ± 0.0022	0.1475
R_e	20.8040 ± 0.0500	20.7440	A_{FB}^b	0.0990 ± 0.0017	0.1034
R_μ	20.7850 ± 0.0330	20.7440	A_{FB}^c	0.0685 ± 0.0034	0.0739
R_τ	20.7640 ± 0.0450	20.7440	A_{LR}	0.1513 ± 0.0021	0.1475
σ_h	41.5410 ± 0.0370	41.4800	M_W	80.450 ± 0.039	80.3890
R_b	0.2165 ± 0.00065	0.2157	M_W/M_Z	0.8822 ± 0.0006	0.8816
R_c	0.1719 ± 0.0031	0.17230	$g_L^2(\nu N \rightarrow \nu X)$	0.3020 ± 0.0019	0.3039
A_{FB}^e	0.0145 ± 0.0025	0.0163	$g_R^2(\nu N \rightarrow \nu X)$	0.0315 ± 0.0016	0.0301
A_{FB}^μ	0.0169 ± 0.0013	0.0163	$g_{eA}(\nu e \rightarrow \nu e)$	-0.5070 ± 0.014	-0.5065
A_{FB}^τ	0.0188 ± 0.0017	0.0163	$g_{eV}(\nu e \rightarrow \nu e)$	-0.040 ± 0.015	-0.0397
$A_\tau(P_\tau)$	0.1439 ± 0.0041	0.1475	$Q_W(Cs)$	-72.65 ± 0.44	-73.11

6.5 Appendix: Predictions for Electroweak Observables

Here we record the shifts to Standard Model values of electroweak observables predicted by the new physics of this model.

$$\begin{aligned}
\Gamma_Z &= (\Gamma_Z)_{SM}(1 - 0.89c_1 + 0.17c_2) \\
R_e &= (R_e)_{SM}(1 + 0.082c_1 + 0.91c_2) \\
R_\mu &= (R_\mu)_{SM}(1 + 0.082c_1 + 0.91c_2) \\
R_\tau &= (R_\tau)_{SM}(1 + 0.082c_1 + 0.91c_2) \\
\sigma_h &= (\sigma_h)_{SM}(1 - 0.0087c_1 - 0.096c_2) \\
R_b &= (R_b)_{SM}(1 - 0.018c_1 - 0.20c_2) \\
R_c &= (R_c)_{SM}(1 + 0.035c_1 + 0.39c_2) \\
A_{FB}^e &= (A_{FB}^e)_{SM} + 0.18c_2 + 2.0c_2 \\
A_{FB}^\mu &= (A_{FB}^\mu)_{SM} + 0.18c_2 + 2.0c_2 \\
A_{FB}^\tau &= (A_{FB}^\tau)_{SM} + 0.18c_2 + 2.0c_2 \\
A_\tau(P_\tau) &= (A_\tau(P_\tau))_{SM} + 0.78c_1 + 8.6c_2 \\
A_e(P_\tau) &= (A_3(P_\tau))_{SM} + 0.78c_1 + 8.6c_2 \\
A_{FB}^p &= (A_{FB}^p)_{SM} + 0.54c_1 + 6.0c_2 \\
A_{FB}^c &= (A_{FB}^c)_{SM} + 0.42c_1 + 4.7c_2 \\
A_{LR} &= (A_{LR})_{SM} + 0.78c_1 + 8.6c_2 \\
M_W &= (M_W)_{SM}(1 + 0.43c_1 + 1.4c_2) \\
M_W/M_Z &= (M_W/M_Z)_{SM}(1 + 0.43c_1 + 1.4c_2) \\
g_L^2(\nu N \rightarrow \nu X) &= (g_L^2(\nu N \rightarrow \nu X))_{SM} + 0.25(c_1 + c_2) \\
g_R^2(\nu N \rightarrow \nu X) &= (g_R^2(\nu N \rightarrow \nu X))_{SM} - 0.085(c_1 + c_2) \\
g_{eV}(\nu e \rightarrow \nu e) &= (g_{eV}(\nu e \rightarrow \nu e))_{SM} - 0.66(c_1 + c_2) \\
g_{eA}(\nu e \rightarrow \nu e) &= (g_{eA}(\nu e \rightarrow \nu e))_{SM} \\
Q_W(Cs) &= (Q_W(Cs))_{SM} + 73(c_1 + c_2)
\end{aligned}$$

Chapter 7

Extension to the $SU(3) \times SU(2) \times U(1)$ Model

The $SU(3) \times SU(2) \times U(1)$ electroweak model described in the preceding chapter suffers from the appearance of exotic stable charged fields. In particular, the $\mathbf{2}_{\pm\mathbf{3}/2}$ states that arise from the $SU(3)$ group are stable and doubly charged. From cosmological observation and heavy ion searches, there are strong constraints on the existence of exotic charged matter. Therefore, these exotic states make the minimal model defined above tightly constrained and an unlikely candidate for the correct theory of nature.

In this chapter we discuss an extension to this model that includes an additional Higgs field along with heavy vector-like states. These additions to the model allow the exotic matter to decay and couple with Standard Model fields, resolving the problem of exotic stable states. However, this extension introduces new parameters in the model, and changes the precision electroweak analysis. We will describe the new model, the decays of exotic states, and present the precision electroweak analysis of the extended model.

7.1 New Matter Fields

In the new model we have an additional Higgs field χ ; the full Higgs sector is given by

$$\Sigma \sim (\mathbf{3}, \bar{\mathbf{2}}_{-1/2}), \quad \langle \Sigma \rangle = \begin{pmatrix} M & 0 \\ 0 & M \\ 0 & 0 \end{pmatrix}; \chi \sim (\mathbf{3}, \mathbf{1}_1), \quad \langle \chi \rangle = \begin{pmatrix} 0 \\ 0 \\ xM \end{pmatrix}. \quad (7.1)$$

These representations and vevs for Σ, χ are deduced by considering the $(\mathbf{3}, \bar{\mathbf{3}})$ field that would be used to break $SU(3) \times SU(3)$ to its diagonal subgroup. Note that Σ, χ fit into a $(\mathbf{3}, \bar{\mathbf{3}})$ representation. The addition of the χ field introduces a new free parameter into the model, namely x . We will assume generally that x is on the order of 1. We also note that by setting $x = 0$, we eliminate the χ field and recover most aspects of the minimal $SU(3) \times SU(2) \times U(1)$ model.

We also introduce some arbitrary number n_F of vector-like fermion pairs

$$\psi_L^i, \psi_R^i \sim \mathbf{3}, \quad i = 1 \dots n_F,$$

with mass M_F . For now we leave n_F as a free parameter and assume that M_F is at or above the symmetry breaking scale M . This vector-like matter will ensure that the exotic gauge fields decay to Standard Model matter.

7.2 Symmetry Breaking

By requiring that the expectation of the Higgs fields be invariant under the action of the gauge group, we have

$$\begin{aligned} \langle \Sigma \rangle &= e^{i\alpha^a t^a} \langle \Sigma \rangle e^{-i\beta^b \tilde{t}^b} e^{-i\delta/2} \\ \langle \chi \rangle &= e^{i\alpha^a t^a} \langle \chi \rangle e^{i\delta}, \end{aligned}$$

and we conclude that

$$A^a = \begin{cases} \beta^a & \text{for } a = 1, \dots, 3 \\ \sqrt{3}\delta & \text{for } a = 8 \\ 0 & \text{otherwise} \end{cases},$$

which is precisely the condition for the diagonal subgroup.

7.3 Gauge Bosons

The results for the gauge boson masses and mass eigenstates are altered slightly by the addition of the χ field, so we present the results from that analysis here.

The Higgs kinetic term in the Lagrangian is extended to include the χ field, yielding

$$\begin{aligned} \mathcal{L} = & \text{Tr} \left[\left(ig_3 A^a t^a \Sigma - i\tilde{g} \tilde{W}^b \tilde{t}^b \Sigma - i\tilde{g}' \frac{B}{2} \Sigma \right)^\dagger \left(ig_3 A^a t^a \Sigma - i\tilde{g} \tilde{W}^b \tilde{t}^b \Sigma - i\tilde{g}' \frac{B}{2} \Sigma \right) \right] \\ & + \text{Tr} \left[(ig_3 A^a t^a \chi + \tilde{g}' B \chi)^\dagger (ig_3 A^a t^a \chi + \tilde{g}' B \chi) \right]. \end{aligned}$$

After substituting vacuum expectation values, we can compute entries of the mass squared matrix. For the $\mathbf{3}_0$ mass eigenstate, the mass squared matrix in the (A^a, \widetilde{W}^a) basis, is

$$M^2 \begin{pmatrix} g_3^2 & -g_3 \tilde{g} \\ -g_3 \tilde{g} & \tilde{g}^2 \end{pmatrix}. \quad (7.2)$$

Therefore, one obtains the mass eigenstates

$$W_L^a = c_\phi A^a - s_\phi \widetilde{W}^a \quad (7.3)$$

$$W_H^a = s_\phi A^a + c_\phi \widetilde{W}^a \quad (7.4)$$

with s_ψ, c_ψ defined as before, and the masses

$$M_{W_L} = 0 \quad (7.5)$$

$$M_{W_H} = (g_3^2 + \tilde{g}^2)^{1/2} M. \quad (7.6)$$

For the $\mathbf{1}_0$ mass eigenstate, the mass squared matrix, in the (A^8, \tilde{B}) basis, is

$$(1 + 2x^2)M^2 \begin{pmatrix} g_3^2/3 & -g_3\tilde{g}'/\sqrt{3} \\ -g_3\tilde{g}'/\sqrt{3} & \tilde{g}'^2 \end{pmatrix}, \quad (7.7)$$

with mass eigenstates

$$B_L = c_\psi A^8 - s_\psi \tilde{B} \quad (7.8)$$

$$B_H = s_\psi A^8 + c_\psi \tilde{B}, \quad (7.9)$$

with s_ϕ, c_ψ defined as before, and the masses

$$M_{B_L} = 0 \quad (7.10)$$

$$M_{B_H} = (1 + 2x^2)^{1/2} \left(\frac{g_3^2}{3} + \tilde{g}'^2 \right)^{1/2} M. \quad (7.11)$$

Finally, the $\mathbf{2}_{\pm 3/2}$ state is formed from the remaining components of the $SU(3)$ adjoint, A^a for $a = 4, 5, 6, 7$. Its mass is given by

$$M_{3/2} = \frac{1}{\sqrt{2}}(1 + x^2)^{1/2} g_3 M, \quad (7.12)$$

where the subscript indicates the hypercharge of the state. The mass eigenstates are obtained from

$$\frac{1}{\sqrt{2}} \begin{pmatrix} A^4 \mp iA^5 \\ A^6 \mp iA^7 \end{pmatrix} \sim \mathbf{2}_{\pm 3/2} \quad (7.13)$$

Again, for $g_3 \ll \tilde{g}, \tilde{g}'$, the $\mathbf{2}_{\pm 3/2}$ gauge bosons will be significantly lighter than the other massive bosons, W_H^a and B_H . The equation for the Standard Model couplings is unaffected, and is given by Eq. (6.17).

7.4 Scalar Potential

Again using the decomposition of $SU(3)$ representations under the $SU(2)_0 \times U(1)_0$ subgroup, we find the following representations under $SU(2)_W \times U(1)_Y$

$$\begin{aligned} \Sigma &\sim (\mathbf{2}_{1/2} + \mathbf{1}_{-1}) \otimes \bar{\mathbf{2}}_{-1/2} = \mathbf{3}_0 + \mathbf{1}_0 + \mathbf{2}_{-3/2} \\ \chi &\sim (\mathbf{2}_{1/2} + \mathbf{1}_{-1}) \otimes \mathbf{1}_1 = \mathbf{2}_{3/2} + \mathbf{1}_0. \end{aligned}$$

Therefore, the full symmetry-breaking sector contains fields with the following representations with complex degrees of freedom

$$\mathbf{3}_0 + \mathbf{1}_0 + \mathbf{1}_0 + \mathbf{2}_{3/2} + \mathbf{2}_{-3/2}..$$

It is generally expected that these scalar fields will obtain masses at the symmetry-breaking scale, M . We study this aspect of the model by constructing the scalar potential and computing local minima. To construct the scalar potential, we list all possible gauge-invariant

operators formed from Σ, χ up to quartic order in the fields. The general approach to obtaining such operators is to look for products of representations that produce singlets under all three gauge groups. We find

$$t_1 = m^2 \text{Tr} \Sigma^\dagger \Sigma, \quad (7.14)$$

$$t_2 = \text{Tr} \Sigma^\dagger \Sigma \text{Tr} \Sigma^\dagger \Sigma, \quad (7.15)$$

$$t_3 = \text{Tr} \Sigma^\dagger \Sigma \Sigma^\dagger \Sigma, \quad (7.16)$$

$$t_4 = \text{Tr} \Sigma \epsilon \Sigma^T \Sigma^* \epsilon \Sigma^\dagger, \quad (7.17)$$

$$t_5 = m^2 \chi^\dagger \chi, \quad (7.18)$$

$$t_6 = \chi^\dagger \chi \chi^\dagger \chi, \quad (7.19)$$

$$t_7 = \chi^\dagger \Sigma \Sigma^\dagger \chi, \quad (7.20)$$

$$t_8 = m \, 2\Re(\Sigma_{i\alpha} \Sigma_{j\beta} \epsilon_{\alpha\beta} \chi_k \epsilon_{ijk}), \quad (7.21)$$

$$t_9 = \chi^\dagger \chi \text{Tr} \Sigma^\dagger \Sigma, \quad (7.22)$$

where m is a mass of the order of the desired symmetry-breaking scale, ϵ_{ijk} is the Levi-Civita symbol, and

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (7.23)$$

The term t_1 can be understood by the fact that all gauge groups here ($SU(3)$, $SU(2)$, $U(1)$), and in fact, all Lie groups, are subsets of unitary matrices. Any element of a Lie group is given by $e^{iA^a T^a}$ where A^a are real coefficients and T^a are the group generators, which are required to be Hermitian. Thus $A^a T^a$ is also Hermitian. A well-known fact about matrix exponentials is that the exponential of iH for H Hermitian gives a unitary matrix. To see this,

$$e^{iH} (e^{iH})^\dagger = e^{iH} e^{-iH^\dagger} = e^{iH} e^{-iH} = \mathbb{I}.$$

We denote arbitrary elements of $SU(3)$, $SU(2)$, $U(1)$ as h_3, h_2, h_1 . Under the action of the full group, we have

$$\begin{aligned} \Sigma &\rightarrow h_3 \Sigma h_2^\dagger \\ \Sigma^\dagger \Sigma &\rightarrow h_2 \Sigma^\dagger h_3^\dagger h_3 \Sigma h_2^\dagger = h_2 \Sigma^\dagger \Sigma h_2^\dagger \\ \text{Tr} \Sigma^\dagger \Sigma &\rightarrow \text{Tr} h_2 \Sigma^\dagger \Sigma h_2^\dagger = \text{Tr} \Sigma^\dagger \Sigma. \end{aligned}$$

In the last step we have used the identity $\text{Tr} S^{-1} A S = \text{Tr} A$ which follows from the cyclic property of Tr .

Therefore, we see that $t_1 = m^2 \text{Tr} \Sigma^\dagger \Sigma$ is invariant under the action of the gauge group. The same conclusion follows immediately for terms t_2, t_3, t_5, t_6, t_9 . The invariance of t_7 follows from a slight variation of this argument.

$$\begin{aligned} \chi &\rightarrow h_3 \chi h_1^\dagger \\ \Sigma &\rightarrow h_3 \Sigma h_2^\dagger \\ \chi^\dagger \Sigma \Sigma^\dagger \chi &\rightarrow h_1 \chi^\dagger h_3^\dagger h_3 \Sigma h_2^\dagger h_2 \Sigma^\dagger h_3^\dagger h_3 \chi h_1^\dagger = \chi^\dagger \Sigma \Sigma^\dagger \chi. \end{aligned}$$

The remaining terms arise from special structure of the $SU(3)$ and $SU(2)$ groups. First we look at t_4 . The group $SU(2)$ has only real representations, so that any representation of $SU(2)$, \mathbf{n} is isomorphic to its complex conjugate representation, $\bar{\mathbf{n}}$. We recall that the complex conjugate representation is given by taking the negative complex conjugates of the original generators, i.e. $-T^a$. To see that this form still satisfies the algebra notice that

$$\begin{aligned} [T^a, T^b] &= if_{abc}T^c \\ [-T^{a*}, -T^{b*}] &= if_{abc}(-T^{c*}) \end{aligned}$$

To see that it still satisfies the group multiplication, consider multiplication of arbitrary group elements

$$\begin{aligned} e^{i\alpha^a T^a} e^{i\beta^b T^b} &= e^{i\chi^c T^c} \\ e^{i\alpha^a (-T^a)^*} e^{i\beta^b (-T^b)^*} &= (e^{i\chi^c T^c})^* = e^{i\chi^c (-T^c)^*} \end{aligned}$$

Actually this fact is quite trivial since $T^a \rightarrow -T^{a*}$ corresponds to a group element $g \rightarrow g^*$. Real representations are ensured to produce gauge theories without anomalies, as shown in [10]. We are now interested in finding the similarity transformation ϵ that relates $\mathbf{2}$ to $\bar{\mathbf{2}}$, as this will allow construction of new invariant operators. So for an arbitrary group element h_2 and $\psi \sim \mathbf{2}$, we should have $h_2 \epsilon \psi = c h_2^* \psi$. Equivalently, $\epsilon^{-1} h_2 \epsilon = h_2^*$. Writing $h_2 = e^{iA^a T^a}$, we find

$$e^{\epsilon^{-1} T^a \epsilon} = e^{i(-T^a)^*} \quad \text{and} \quad \epsilon^{-1} T^a \epsilon = -T^{a*}.$$

The matrix c can now easily be deduced from the form of the $SU(2)$ generators, and we obtain

$$\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now we are equipped to construct singlets from combinations of $\mathbf{2}, \bar{\mathbf{2}}$ representations. Notice that $\Sigma^* \sim \bar{\mathbf{2}}$. Therefore

$$\begin{aligned} \Sigma &\rightarrow h_3 \Sigma h_2^\dagger \\ \text{Tr} \Sigma \epsilon \Sigma^T \Sigma^* \epsilon \Sigma^\dagger &\rightarrow \text{Tr} h_3 \Sigma h_2^\dagger \epsilon h_2^* \Sigma^T h_3^T h_3^* \Sigma^* h_2^T \epsilon h_2 \Sigma^\dagger h_3^\dagger = \text{Tr} \Sigma \epsilon \Sigma^T \Sigma^* \epsilon \Sigma^\dagger. \end{aligned}$$

Now we consider t_8 . This term is derived from the fact that $\mathbf{3} \times \mathbf{3} \times \mathbf{3}$ contains a singlet. The Clebsch-Gordon coefficients that yield a singlet from the product of three triplets are given by the Levi-Civita symbol ϵ_{ijk} . Consider a field ψ transforming under $SU(3)$ as $\psi \sim \mathbf{3}$. Then the term

$$\epsilon_{ijk} \psi_i \psi_j \psi_k,$$

is invariant under $SU(3)$. We can check infinitesimal transformations, where $h_3 = e^{i da t^a}$ is an infinitesimal group element. We expand the exponential as a Taylor series, and, since it is infinitesimal, we keep only linear terms, i.e. $h_3 = 1 + i da t^a$. Looking at the invariant term, we see

$$\begin{aligned} \epsilon_{ijk} \psi_i \psi_j \psi_k &\rightarrow \epsilon_{ijk} [(1 + i da t^a) \psi]_i [(1 + i da t^a) \psi]_j [(1 + i da t^a) \psi]_k \\ &= \epsilon_{ijk} \psi_i \psi_j \psi_k + \epsilon_{ijk} da ((t^a \psi)_i \psi_j \psi_k + \psi_i (t^a \psi)_j \psi_k + \psi_i \psi_j (t^a \psi)_k) \\ &= \epsilon_{ijk} \psi_i \psi_j \psi_k + \epsilon_{ijk} da ((t^a)_{i,i'} \psi_{i'} \psi_j \psi_k + \psi_i (t^a)_{j,j'} \psi_{j'} \psi_k + \psi_i \psi_j (t^a)_{k,k'} \psi_{k'}) \end{aligned}$$

The invariance of this term is equivalent to second summand in the last expression being equal to zero. By rearranging the primed dummy indices, we obtain the following condition which is easily checked computationally

$$\epsilon_{i'jk}(t^a)_{i'i} + \epsilon_{ij'k}(t^a)_{j'j} + \epsilon_{ijk'}(t^a)_{k'k} = 0. \quad (7.24)$$

The t_8 term combines two copies of Σ and one χ field, all of which transform as $\mathbf{3}$, into a singlet. The term must also be invariant under the $SU(2)$, so the $SU(2)$ parts of the Σ field are combined with coefficients from the c matrix. We take the real part since all terms in the potential must be real.

The potential is an arbitrary linear combination of these invariant terms, with coefficients αI ,

$$V = \sum_{i=1}^9 \alpha_i t_i. \quad (7.25)$$

The terms t_1, t_5 are always given negative coefficients to ensure that there is no local minimum at the origin. To find a local minimum, we perform a constrained minimization in which we assume that the vacuum expectation values are of the form in Eq. (7.1). Substituting these values into the potential, we minimize the resulting function V_0 , given by

$$V_0 = (2\alpha_1 + \alpha_5 x^2) m^2 M^2 + (4\alpha_2 + 2\alpha_3 - 2\alpha_4 + 2\alpha_9 x^2 + \alpha_6 x^4) M^4 + 4\alpha_8 x m M^3. \quad (7.26)$$

For an example, setting $(\alpha_1, \dots, \alpha_9) = (-1, 1.1, 1.2, 1.4, -1.3, 0.9, 0.7, 0.8, 0.5)$, we find that the global minimum of the potential over the parameters M and x is at

$$(M, x) = (0.720 m, 1.33).$$

Note that m can be eliminated from the expression for V_0 by dividing by m^4 and minimizing over a dimensionless parameter M/m that give M in units of m . We confirm that this point is a minimum by checking positive definiteness of the scalar mass squared matrix

$$M_{ij}^2 = \frac{1}{2} \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}, \quad (7.27)$$

where the ϕ_i denote the real scalar degrees of freedom in the fields Σ and χ , with $1 \leq i \leq 18$. This mass matrix makes sense because masses of scalar fields appear in the potential as $\frac{1}{2} m^2 \phi^2$. Therefore, the massive scalar states are appropriate linear combinations of real scalar degrees of freedom that form eigenstates of the mass matrix. The squared masses are all positive, producing physical masses as shown in Table 7.1, and have the correct multiplicity to occupy complete representations of the unbroken gauge group. For this choice of parameters α_i , we also confirm that there are eight zero eigenvalues, corresponding precisely to the $12 - 4$ broken generators in the spontaneous breaking $SU(3) \times SU(2) \times U(1) \rightarrow SU(2)_W \times U(1)_Y$. The zero eigenvalues represent the eaten $\mathbf{3}_0, \mathbf{1}_0, \mathbf{2}_{\pm 3/2}$ states. For every choice of parameters we have tried, we have found similar results, indicating that there is no difficulty in locating parameters that produce a viable potential.

Table 7.1: Spectrum of physical scalars in the $SU(3)$ model, in units of m , for the example parameter choice described in the text. The states listed here have only real degrees of freedom.

state	multiplicity	mass
$\mathbf{3}_0$	3	2.06
$\mathbf{2}_{\pm 3/2}$	4	1.30
$\mathbf{1}_0$	1	1.55
$\mathbf{1}_0$	1	1.40
$\mathbf{1}_0$	1	1.32

7.5 Electroweak Constraints

We are now interested in analyzing the electroweak constraints on the extended $SU(3)$ model. There are three primary differences from the minimal $SU(3)$ model, which alter the analysis. First, the low-energy Lagrangian will depend on the masses of the heavy W_H, B_H fields, which have changed due to the introduction of the χ field and depend on the parameter x . Second, the new matter in this model, the χ field and the n_F vector-like pairs, alter the beta functions. Third, we use updated experimental data for electroweak observables from LEP II and a slightly different computation of the Standard Model predictions.

By following the analysis done in [4], we note that the theoretical predictions for electroweak observables in this model are obtained from the minimal $SU(3)$ results via the substitution

$$c_\psi^4 \rightarrow \frac{c_\psi^4}{1 + 2x^2}.$$

Therefore, the corrections to Standard Model values for the minimal $SU(3)$ model can be carried over by defining

$$c_1 = \frac{c_\phi^4 v^2}{4M^2} \quad \text{and} \quad c_2 = \frac{1}{1 + 2x^2} \frac{c_\psi^4 v^2}{4M^2}.$$

We refer to Sec. 6.5 for the corrections predicted by the minimal $SU(3)$ model. Following the approach of Ref. [4], we construct a chi-squared function for the shifts in electroweak precision observables from their Standard Model values as a function of the parameters c_1 and c_2 . The Standard Model predictions and experimental data are taken from a fit by Langacker and Erler that appears in the 2006 Review of Particle Properties [8]. For convenience, we quote these values in Table 7.2. The best fit in c_1, c_2 has a chi-squared value of $X_{min} = 31.5$. The main difference in the electroweak data that we use in for our fit compared to Ref. [4] is that more recent LEP II results have shifted the central value of the W mass downward. Since the nonstandard contribution to M_W in our model is positive, the parameter space is now more tightly constrained. We illustrate this in Fig. 7.5, which displays the 68%, 95% and 99% confidence contours from our global fit compared to those in Ref. [4]. The shift in these contours does not lead to a dramatic change in the allowed parameter space of the

Table 7.2: Input parameters for the electroweak fit described in the text. The SM column shows central values from a Standard Model fit by Langacker and Erler, appearing in the 2006 Review of Particle Physics [8], in which $m_Z = 91.1874 \pm 0.0021$ GeV, $m_H = 89_{-28}^{+38}$ GeV, $m_t = 172.7 \pm 2.8$ GeV and $\alpha_s(m_Z) = 0.1216 \pm 0.0017$.

Quantity	Experiment	SM	Quantity	Experiment	SM
Γ_Z	2.4952 ± 0.0023	2.4968	$A_e(P_\tau)$	0.1498 ± 0.0049	0.1471
R_e	20.8040 ± 0.0500	20.7560	A_{FB}^b	0.0992 ± 0.0016	0.1031
R_μ	20.7850 ± 0.0330	20.7560	A_{FB}^c	0.0707 ± 0.0035	0.0737
R_τ	20.7640 ± 0.0450	20.8010	A_{LR}	0.15138 ± 0.00216	0.1471
σ_h	41.5410 ± 0.0370	41.4670	M_W	80.403 ± 0.029	80.3760
R_b	0.21629 ± 0.00066	0.21578	M_W/M_Z	0.88173 ± 0.00032	0.8814
R_c	0.1721 ± 0.0030	0.17230	$g_L^2(\nu N \rightarrow \nu X)$	0.30005 ± 0.00137	0.30378
A_{FB}^e	0.0145 ± 0.0025	0.01622	$g_R^2(\nu N \rightarrow \nu X)$	0.03076 ± 0.00110	0.03006
A_{FB}^μ	0.0169 ± 0.0013	0.01622	$g_{eA}(\nu e \rightarrow \nu e)$	-0.5070 ± 0.014	-0.5064
A_{FB}^τ	0.0188 ± 0.0017	0.01622	$g_{eV}(\nu e \rightarrow \nu e)$	-0.040 ± 0.015	-0.0396
$A_\tau(P_\tau)$	0.1439 ± 0.0043	0.1471	$Q_W(Cs)$	-72.62 ± 0.46	-73.17

minimal SU(3) model. However, the shape of the exclusion region in our models depends noticeably on the value of the parameter x , as we describe below.

The parameter space of the model may be described in terms of the couplings \tilde{g}' , and \tilde{g} . As in Ref. [4], we define the unification scale M_U as the mass of the heaviest gauge boson, the threshold at which the matching conditions Eq. (6.17) should be applied. The Standard Model gauge couplings $g(M_U)$ and $g'(M_U)$, are determined via the one-loop renormalization group equations

$$\alpha^{-1}(M_U) = \alpha^{-1}(M_Z) + \frac{b_{SM}}{2\pi} \ln \frac{M_U}{M_Z} + \sum_i \frac{b_i}{2\pi} \ln \frac{M_U}{M_i} , \quad (7.28)$$

$$\alpha'^{-1}(M_U) = \alpha'^{-1}(M_Z) + \frac{b'_{SM}}{2\pi} \ln \frac{M_U}{M_Z} + \sum_i \frac{b'_i}{2\pi} \ln \frac{M_U}{M_i} , \quad (7.29)$$

where M_i is the mass of the i^{th} heavy particle threshold, and b_i the contribution to the beta function. For the heavy gauge bosons, the M_i are proportional to M_U , since the unification scale is identified as $\max\{M_{W_H}, M_{Z_H}\}$; the other heavy boson states are always lighter than this result. The physical scalar components of the Σ and χ Higgs fields are taken to have the same mass as the $\mathbf{2}_{\pm 3/2}$ gauge bosons, the same approximation used in Ref. [4]. The mass scale of the vector-like matter is separately specified as M_F . The values for the beta functions are given in Table 7.3. If one specifies \tilde{g}' and \tilde{g} , then Eqs. (6.17), (7.28) and 7.29 completely determine M_U and the coupling $g_3(M_U)$. The quantities c_ϕ and c_ψ follow immediately, while the parameter M is known through the identification of M_U with the heaviest gauge boson mass. All the quantities needed to compute the values of c_1 and c_2 are thereby obtained. We implement this procedure numerically to associate each point in the

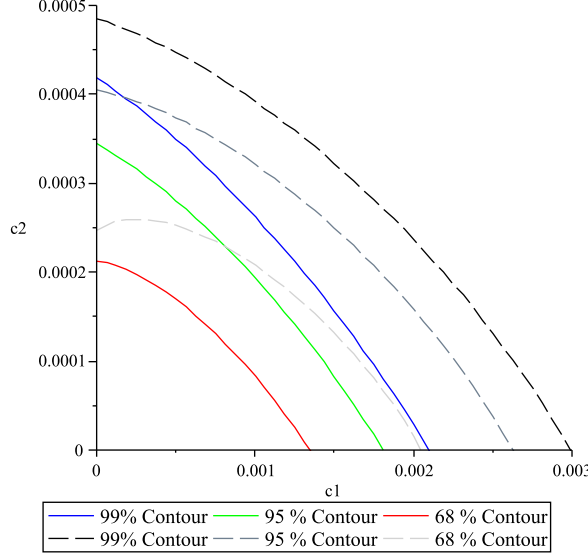


Figure 7.1: Confidence contours in $c_1 - c_2$ space. The new contours are given in solid, colored lines, whereas the contours of the minimal $SU(3)$ model are given in dashed, grey lines.

states	b_i	b'_i
Standard Model	$19/6$	$-41/6$
$\mathbf{2}_{\pm 3/2}$ vector	$7/2$	$63/2$
physical scalars	$-1/2$	$-3/2$
vector-like	$-2n_F/3$	$-2n_F$

Table 7.3: Beta functions b_i in Eqs. (7.28) and (7.29). Vector boson beta functions include the contribution from the longitudinal (eaten scalar) component. Physical scalars are assumed to have the same mass as the $\mathbf{2}_{3/2}$ vector bosons. The number of $\mathbf{3} + \bar{\mathbf{3}}$ pairs is given by n_F .

\tilde{g}' - \tilde{g} plane with a point in c_1 - c_2 space; in this way, we determine whether a given point in the model's parameter space is excluded, to any desired confidence level. We show the 95% confidence level exclusion regions in the results that follow. To complete the analysis, we must specify values of x and the number of $\mathbf{3} + \bar{\mathbf{3}}$ pairs n_F . In the appendix we provide contour plots for the matching scale M_U and the lightest gauge boson mass assuming $M_F = 1$ TeV, and allowing $x \in \{0, 1/3, 1, 3\}$ and $n_F \in \{0, 1, 3\}$.

Larger values of x tends to exclude smaller values of \tilde{g} ; however, the constant M_U contours and the boundary of the excluded region move in tandem, so that the effect on the smallest allowed value of M_U is relatively mild. It is worth noting that there is an optimal choice $x \approx 1.2$ for which the $M_U = 10$ TeV contour is within the allowed region for $\tilde{g}' < 1.2$ and $\tilde{g} < 2$, an improvement over the minimal $SU(3)$ model. We also include plots for this optimal value of x . However, at large values, $x \geq 3$, the exclusion region grows, engulfing the entire $M_U = 10$ TeV contour. Varying the number of heavy fermion pairs between 0 and 3 has a

negligible effect on the position of these contours or the excluded region. We also plot mass contours for the $\mathbf{2}_{3/2}$ gauge boson in the $SU(3)$ model, with $n_F = 0$. Generally, we note that $M_{3/2} = 2$ TeV is entirely excluded and $M_{3/2} = 4$ TeV entirely outside the exclusion region. For an optimal value of $x \approx 0.9$, the $M_{3/2} = 3$ TeV curve is completely outside the exclusion region. Again, for $x \geq 3$, the exclusion region becomes large, excluding $M_{3/2} = 4$ TeV as well.

7.6 Decays of Exotic States

The motivation of extending the $SU(3)$ model was to circumvent the cosmological difficulties of exotic, stable, charged matter. Therefore, we wish to demonstrate that this extension of the model allows the previously stable gauge bosons to decay to Standard Model matter. The Lagrangian term that involves the new vector-like matter is

$$\mathcal{L} = \bar{\psi}(i \not{D} - M_F)\psi - [\bar{\psi}_L \Sigma \lambda^\ell \ell_R + \bar{\psi}_L \chi \lambda^e e_R + \text{h.c.}] , \quad (7.30)$$

where ℓ_R and e_R are the Standard Model $SU(2)_W$ doublet and singlet leptons, and λ^ℓ and λ^e are Yukawa matrices that mix the heavy and light states. This operator in the Lagrangian allows the exotic $SU(3)$ gauge bosons to decay ultimately to Standard Model leptons. Because we set the scale of the vector-like matter $M_F > M_U$, we can integrate it out to obtain an effective Lagrangian. The equation of motion from the Lagrangian term in Eq. (7.30) yields

$$\psi = -\frac{1}{M_F} \left(1 + \frac{i \not{D}}{M_F} \right) [\Sigma \lambda^\ell \ell_R + \chi \lambda^e e_R] , \quad (7.31)$$

where we keep only terms up to first order in \not{D}/M_F . Substituting back into the Lagrangian term, we find the effective Lagrangian, to first order in $1/M_F^2$,

$$\mathcal{L}_{eff} = \frac{1}{M_F^2} (\bar{e}_R \lambda^{e\dagger} \chi^\dagger) i \not{D} (\Sigma \lambda^\ell \ell_R) + \frac{1}{M_F^2} (\bar{e}_R \lambda^{e\dagger} \chi^\dagger) i \not{D} (\chi \lambda^e e_R) + \text{h.c.} . \quad (7.32)$$

The relevant gauge bosons here are given by Eq. (7.13)

$$A_{\pm 3/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} A^4 \mp i A^5 \\ A^6 \mp i A^7 \end{pmatrix} \sim \mathbf{2}_{\pm 3/2}.$$

Applying this to the effective Lagrangian term, we get the X-fermion-fermion coupling,

$$\mathcal{L}_{A_{\pm 3/2}} = g_3 x \left(\frac{M}{M_F} \right)^2 \bar{e}_R [\lambda^{e\dagger} A_{\pm 3/2}^\dagger \lambda^\ell] \ell_R + \text{h.c.} . \quad (7.33)$$

We note that the coupling vanishes in the limits $M_F \rightarrow \infty$ or $x \rightarrow 0$, which correspond to removing the vector-like pairs and the χ field from the theory, respectively. Therefore both fields are necessary to facilitate the coupling and decay.

The decay process of the $\mathbf{2}_{\pm 3/2}$ state is especially interesting, as it contains a doubly-charged bilepton field. Computation of the decay time gives

$$c \tau_X = 0.007 \text{ cm} \cdot \left(\frac{3 \text{ TeV}}{M_{3/2}} \right)^5 \cdot \left(\frac{M_F}{10 \text{ TeV}} \right)^4 \cdot \left[\frac{g_3^2 (1+x^2)^2}{4x^2} \right] , \quad (7.34)$$

where we consider only the mixing with one Standard Model generation, and take $\lambda^e = \lambda^\ell = \sqrt{2}m_e/v$. This state can be arbitrarily long-lived, depending on the value of M_F which is unconstrained by the low energy theory. With three generations, these couplings could allow flavor-violating decays. Thus this process would produce an interesting signature in particle colliders, and further study of the collider physics involved may be worthwhile.

7.7 Appendix: Plots of Unification and Gauge Boson Mass Contours

Here we include the full collection of plots for the unification contours and contours of lightest boson mass for the extended $SU(3)$ model. Note that the typesetting on the plots omits tildes from the axes labels. All the plots are actually in $\tilde{g} - \tilde{g}'$ space.

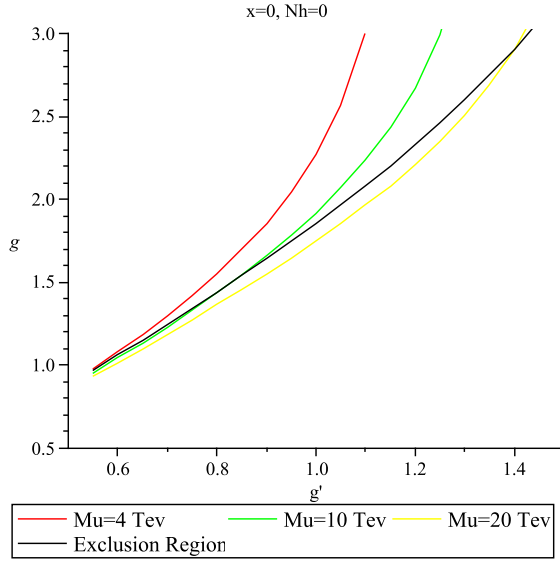


Figure 7.2: Contours of the unification scale M_U for $n_F = 0$, $x = 0$. The region above the black line is excluded at the 95% confidence level.

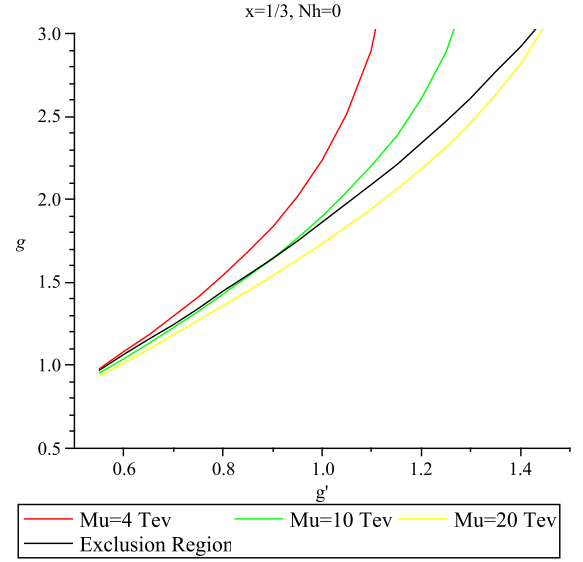


Figure 7.3: Contours of the unification scale M_U for $n_F = 0$, $x = 1/3$. The region above the black line is excluded at the 95% confidence level.

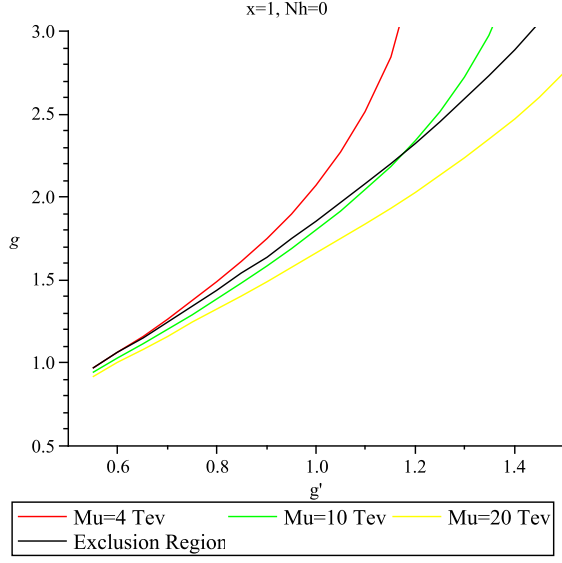


Figure 7.4: Contours of the unification scale M_U for $n_F = 0$, $x = 1$. The region above the black line is excluded at the 95% confidence level.

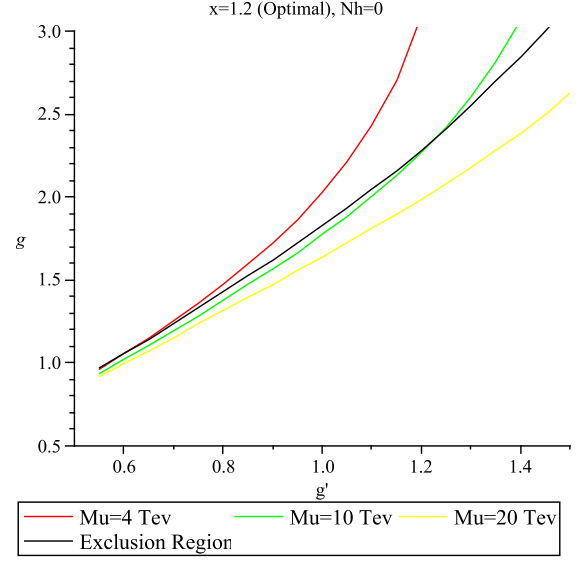


Figure 7.5: Contours of the unification scale M_U for $n_F = 0$, at the optimal value of $x = 1.2$. The region above the black line is excluded at the 95% confidence level.

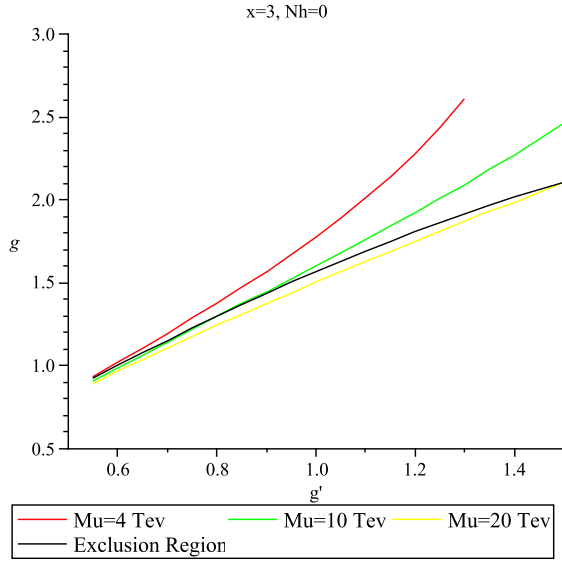


Figure 7.6: Contours of the unification scale M_U for $n_F = 0$, $x = 3$. The region above the black line is excluded at the 95% confidence level.

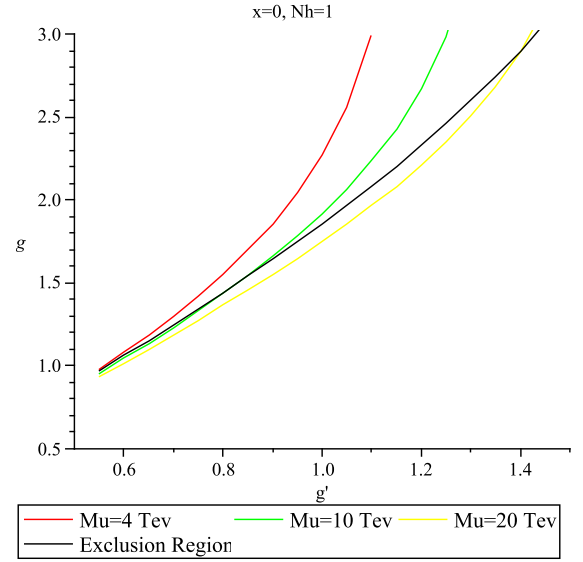


Figure 7.7: Contours of the unification scale M_U for $n_F = 1$, $x = 0$. The region above the black line is excluded at the 95% confidence level.

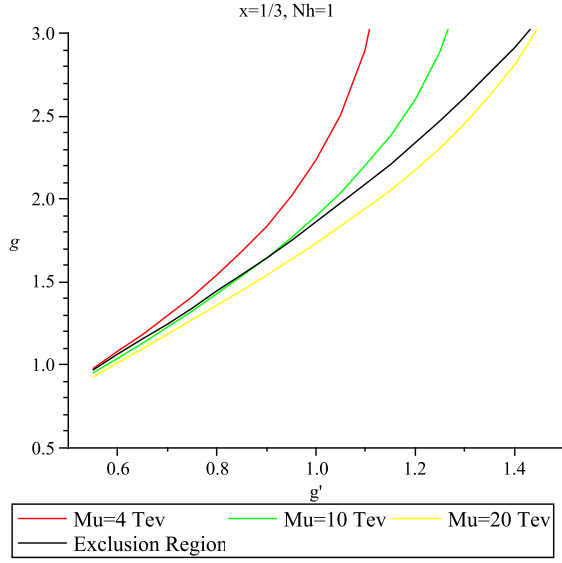


Figure 7.8: Contours of the unification scale M_U for $n_F = 1$, $x = 1/3$. The region above the black line is excluded at the 95% confidence level.

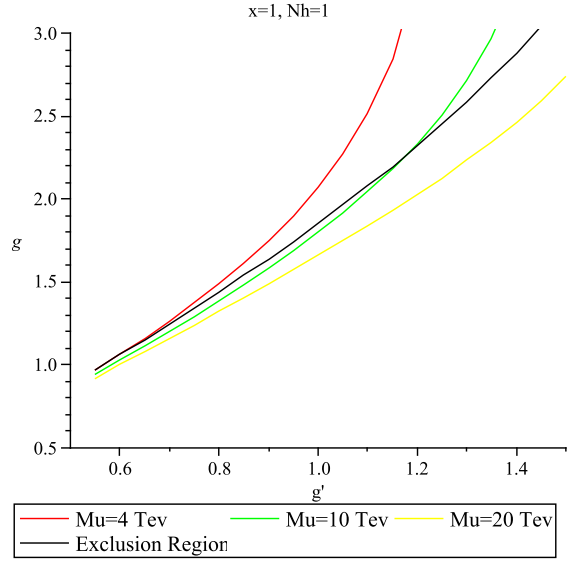


Figure 7.9: Contours of the unification scale M_U for $n_F = 1$, $x = 1$. The region above the black line is excluded at the 95% confidence level.

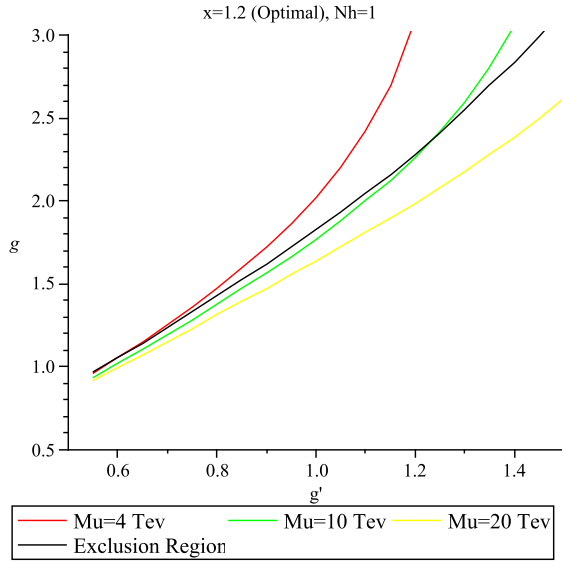


Figure 7.10: Contours of the unification scale M_U for $n_F = 1$, at the optimal value of $x = 1.2$. The region above the black line is excluded at the 95% confidence level.

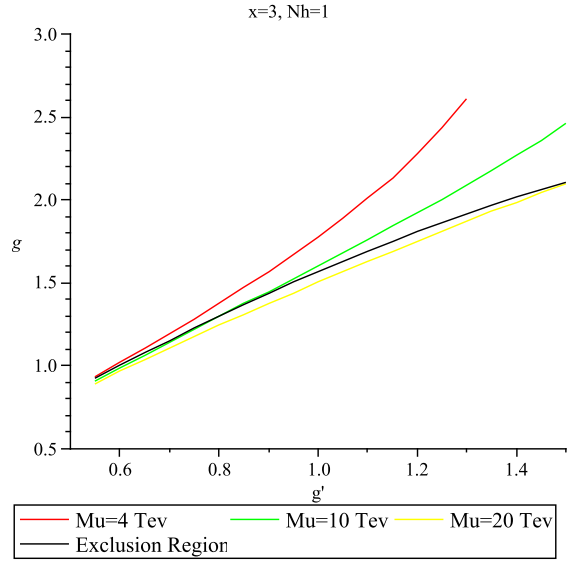


Figure 7.11: Contours of the unification scale M_U for $n_F = 1$, $x = 3$. The region above the black line is excluded at the 95% confidence level.

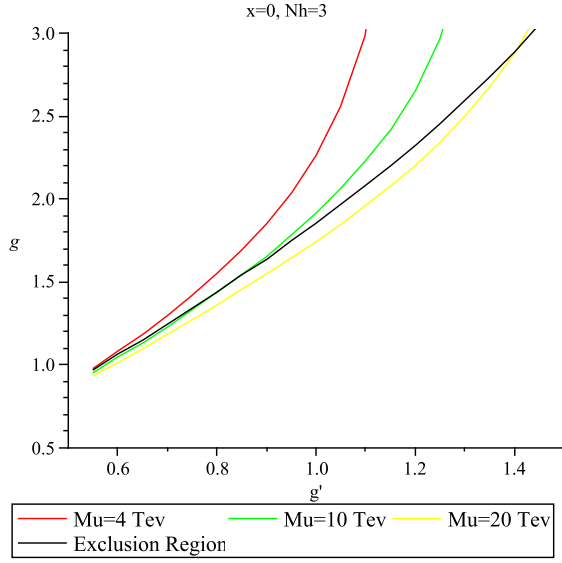


Figure 7.12: Contours of the unification scale M_U for $n_F = 3$, $x = 0$. The region above the black line is excluded at the 95% confidence level.

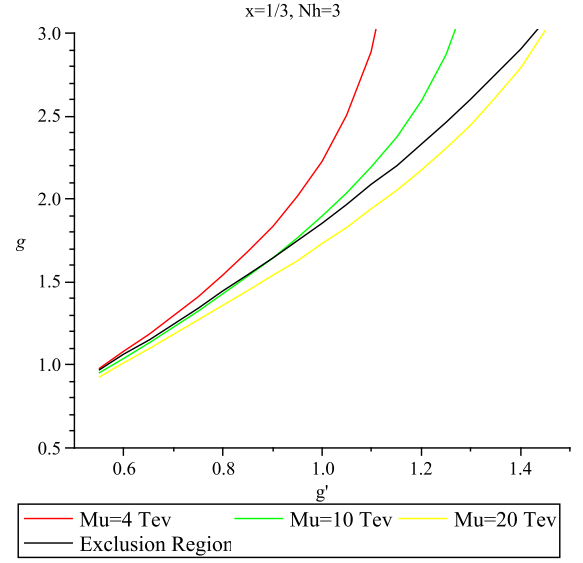


Figure 7.13: Contours of the unification scale M_U for $n_F = 3$, $x = 1/3$. The region above the black line is excluded at the 95% confidence level.

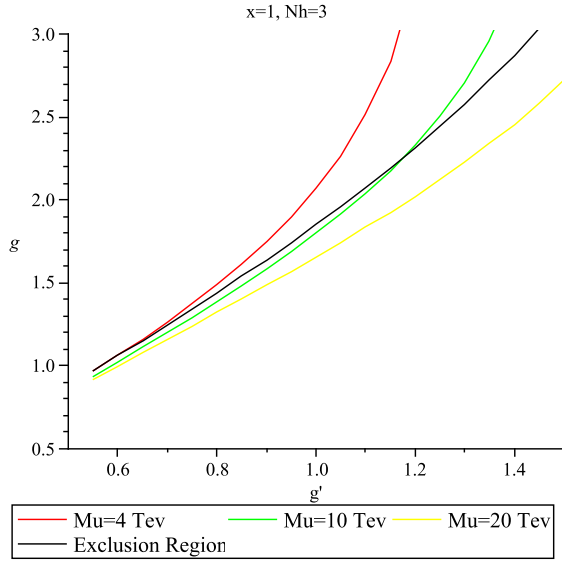


Figure 7.14: Contours of the unification scale M_U for $n_F = 3$, $x = 1$. The region above the black line is excluded at the 95% confidence level.

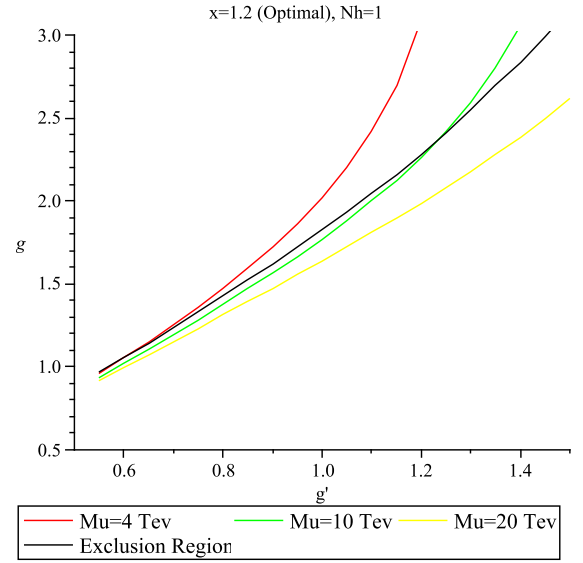


Figure 7.15: Contours of the unification scale M_U for $n_F = 3$, at the optimal value of $x = 1.2$. The region above the black line is excluded at the 95% confidence level.

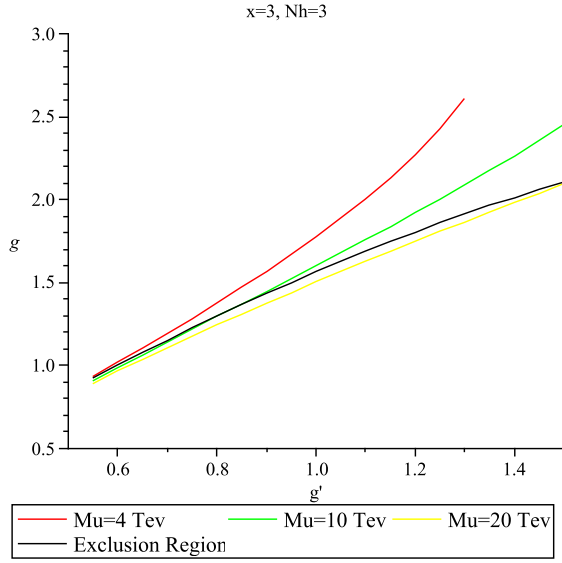


Figure 7.16: Contours of the unification scale M_U for $n_F = 3$, $x = 3$. The region above the black line is excluded at the 95% confidence level.

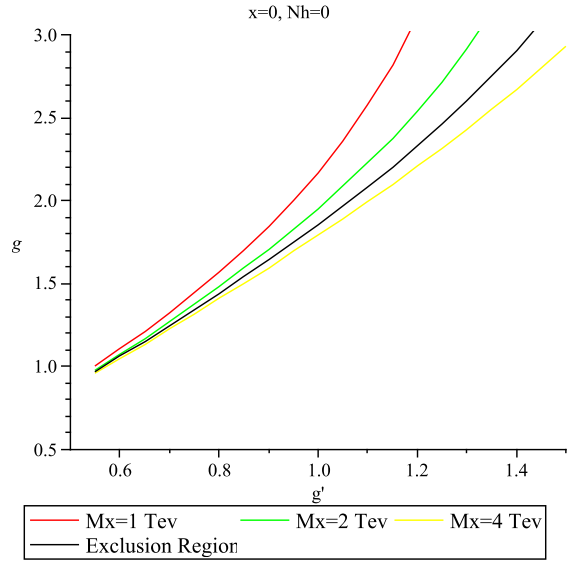


Figure 7.17: Contours of the mass of the lightest gauge boson, $2_{\pm 3/2}$ for $n_F = 0$, $x = 0$. The region above the black line is excluded at the 95% confidence level.

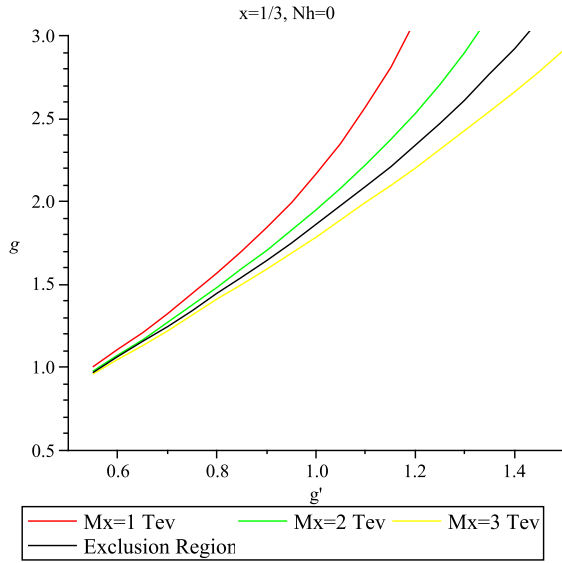


Figure 7.18: Contours of the mass of the lightest gauge boson, $2_{\pm 3/2}$ for $n_F = 0$, $x = 1/3$. The region above the black line is excluded at the 95% confidence level.

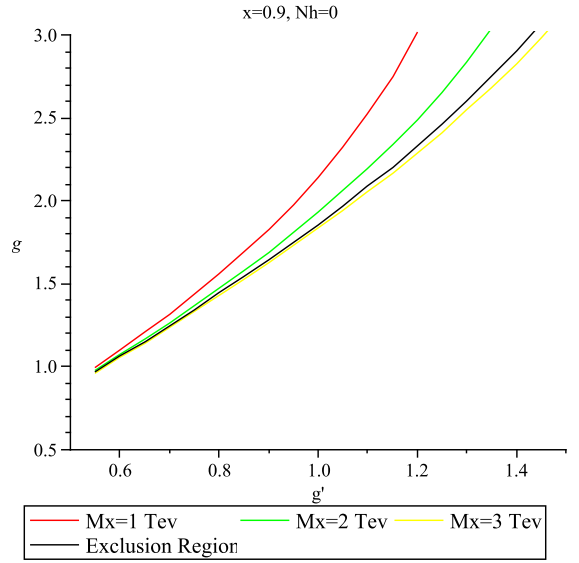


Figure 7.19: Contours of the mass of the lightest gauge boson, $2_{\pm 3/2}$ for $n_F = 0$, at the optimal value of $x = 0.9$. The region above the black line is excluded at the 95% confidence level.

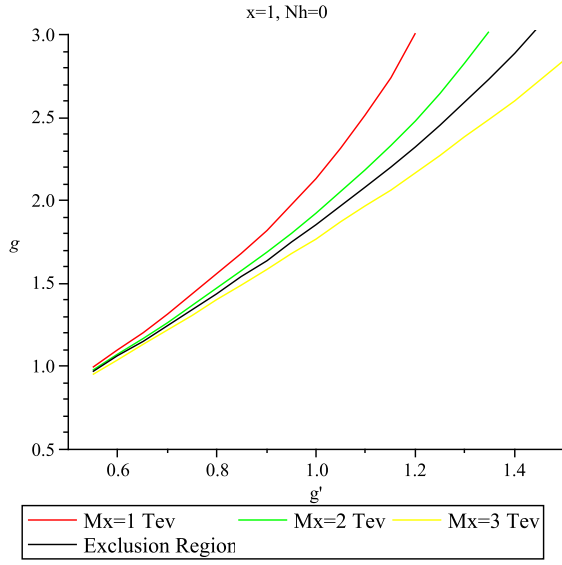


Figure 7.20: Contours of the mass of the lightest gauge boson, $\mathbf{2}_{\pm 3/2}$ for $n_F = 0$, $x = 1$. The region above the black line is excluded at the 95% confidence level.

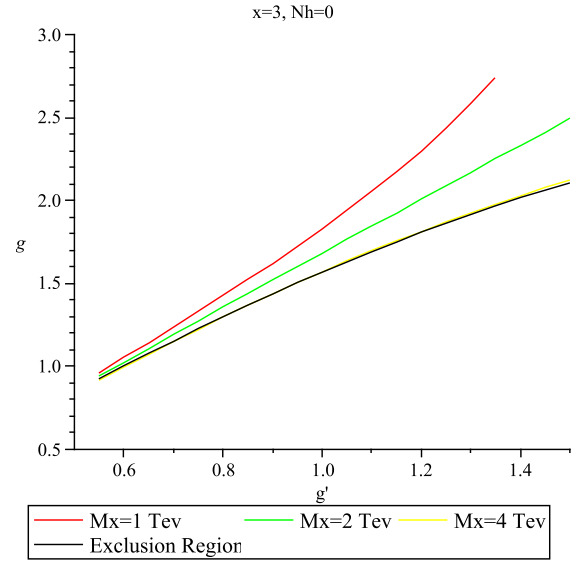


Figure 7.21: Contours of the mass of the lightest gauge boson, $\mathbf{2}_{\pm 3/2}$ for $n_F = 0$, $x = 3$. The region above the black line is excluded at the 95% confidence level.

Chapter 8

G_2 Electroweak Model

As an alternative to introducing new matter to the minimal $SU(3)$ electroweak model, one might consider embedding the $SU(3)$ group into a larger Lie group. Of course, the difficulty in extending the model this way is that it results in more exotic gauge bosons which must be made to decay to Standard Model matter. Therefore, in taking this approach, we wish to use the ‘smallest’ Lie group available, into which we can embed the $SU(3)$. Since $SU(3)$ is a rank 2 group, any such group must be at least rank 2. As it turns out, G_2 is the only rank 2 group that properly contains $SU(3)$. Furthermore, G_2 has fewer generators than any other group that properly contains $SU(3)$. Therefore, the extension to the gauge group $G_2 \times SU(2) \times U(1)$ represents a next-to-minimal model.

8.1 Symmetry breaking

The Higgs sector and symmetry breaking follow in exact analogy to the extended $SU(3)$ model described in Ch. 7. We briefly describe these elements. The Higgs sector is given by naturally embedding the Σ, χ fields into representations of G_2 . We have

$$\Sigma \sim (\mathbf{7}, \bar{\mathbf{2}}_{-1/2}), \quad \langle \Sigma \rangle = \begin{pmatrix} M & 0 \\ 0 & M \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \chi \sim (\mathbf{7}, \mathbf{1}_1), \quad \langle \chi \rangle = \begin{pmatrix} 0 \\ 0 \\ x M \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (8.1)$$

Again, these fields are naturally obtained from the $(\mathbf{7}, \bar{\mathbf{7}})$ field that would break $G_2 \times G_2$ to its diagonal subgroup. The symmetry breaking follows in a similar way. By requiring that the expectation of the Higgs fields be invariant under the action of the gauge group, we have

$$\begin{aligned} \langle \Sigma \rangle &= e^{i\alpha^a T^a} \langle \Sigma \rangle e^{-i\beta^b \bar{T}^b} e^{-i\delta/2} \\ \langle \chi \rangle &= e^{i\alpha^a T^a} \langle \chi \rangle e^{i\delta}, \end{aligned}$$

and we conclude that

$$A^a = \begin{cases} \sqrt{6}\beta^a & \text{for } a = 1, \dots, 3 \\ \sqrt{3}\delta & \text{for } a = 8 \\ 0 & \text{otherwise} \end{cases},$$

which is precisely the diagonal subgroup.

From our study in Sec. 5.3, we know that the fundamental representation of G_2 decomposes under $SU(3)$ as

$$\mathbf{14} = \mathbf{8} + \mathbf{3} + \bar{\mathbf{3}}.$$

Therefore the representations of the gauge bosons under $SU(2)_W \times U(1)_Y$ are given by

$$\mathbf{14} = (\mathbf{3}_0 + \mathbf{1}_0 + \mathbf{2}_{3/2} + \mathbf{2}_{-3/2}) + (\mathbf{2}_{1/2} + \mathbf{1}_{-1}) + (\mathbf{2}_{-1/2} + \mathbf{1}_1).$$

Pairs of representations with the same $SU(2)_W$ representations and opposite hypercharges will appear as complex vector fields.

The mass spectrum of the gauge bosons in the $SU(3) \times SU(2) \times U(1)$ subgroup, that is, W_H , W_L , B_H , B_L , and the exotic $\mathbf{2}_{\pm 3/2}$ states, as well as the mixing angles ϕ and ψ , are precisely the same as in the $SU(3)$ model of Ch. 7, with the substitution

$$g_3 = g_2/\sqrt{2}, \quad (8.2)$$

where g_2 is the new G_2 gauge coupling. The $\sqrt{2}$ factor arises from the relative normalizations of the group generators. We find that the masses of the new $\mathbf{2}_{\pm 1/2}$, $\mathbf{1}_{\pm 1}$ bosons are given by

$$M_{1/2} = \frac{1}{\sqrt{6}}(3 + x^2)^{1/2} g_3 M, \quad (8.3)$$

$$M_1 = \frac{1}{\sqrt{3}}(1 + x^2)^{1/2} g_3 M, \quad (8.4)$$

where the subscripts again refer to the hypercharges of the states. These mass eigenstates are again obtained from analysis in Sec. 5.4; we have

$$A_{\mathbf{2}_{1/2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} A^9 \pm iA^{10} \\ A^{11} \pm iA^{12} \end{pmatrix} \sim \mathbf{2}_{\pm 1/2}, \quad A_{\mathbf{1}_1} \frac{1}{\sqrt{2}}(A^{13} \pm iA^{14}) \sim \mathbf{1}_{\mp 1}.$$

Note that the $\mathbf{1}_{\pm 1}$ states are always lighter than the $\mathbf{2}_{\pm 3/2}$ bosons of the $SU(3)$ model, indicating that this model may be a better candidate for detection at particle colliders. In this model, we exclude the vector-like $\mathbf{3} + \bar{\mathbf{3}}$ pairs that were introduced in the extended $SU(3)$ model. Although these fields allow decay of the $SU(3)$ gauge bosons contained in G_2 , it will be more straightforward to handle the G_2 decays in an alternative approach.

8.2 Scalar Potential

We now consider the scalar sector of the model. We recall that the fundamental representation of G_2 decomposes under $SU(3)$ as

$$\mathbf{7} = \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1},$$

and deduce that the Σ and χ fields contain the following $SU(2)_W \times U(1)_Y$ representations:

$$\Sigma = (\mathbf{3}_0 + \mathbf{1}_0 + \mathbf{2}_{-3/2}) + (\mathbf{3}_{-1} + \mathbf{1}_{-1} + \mathbf{2}_{1/2}) + \mathbf{2}_{-1/2} , \quad (8.5)$$

$$\chi = (\mathbf{2}_{3/2} + \mathbf{1}_0) + (\mathbf{2}_{1/2} + \mathbf{1}_2) + \mathbf{1}_1 . \quad (8.6)$$

We will follow the previous approach to demonstrate local minima of an appropriately invariant scalar potential with the desired pattern of symmetry-breaking vevs given in Eq. (8.1). The invariant terms up to quartic order in the Higgs fields are given as

$$u_1 = m^2 \text{Tr} \Sigma^\dagger \Sigma \quad (8.7)$$

$$u_2 = \text{Tr} \Sigma^\dagger \Sigma \text{Tr} \Sigma^\dagger \Sigma \quad (8.8)$$

$$u_3 = \text{Tr} \Sigma^\dagger \Sigma \Sigma^\dagger \Sigma \quad (8.9)$$

$$u_4 = \text{Tr} \Sigma^T S \Sigma \Sigma^\dagger S \Sigma^* \quad (8.10)$$

$$u_5 = \text{Tr} \Sigma \epsilon \Sigma^T \Sigma^* \epsilon \Sigma^\dagger \quad (8.11)$$

$$u_6 = \text{Tr} \Sigma^T S \Sigma \epsilon \text{Tr} \Sigma^\dagger S \Sigma^* \epsilon \quad (8.12)$$

$$u_7 = m^2 \chi^\dagger \chi \quad (8.13)$$

$$u_8 = \chi^\dagger \chi \chi^\dagger \chi \quad (8.14)$$

$$u_9 = \chi^T S \chi \chi^\dagger S \chi^* \quad (8.15)$$

$$u_{10} = m \, 2\Re \Sigma_{i\alpha} \Sigma_{j\beta} \epsilon_{\alpha\beta} \chi_k C_{ijk} \quad (8.16)$$

$$u_{11} = \chi^\dagger \Sigma \Sigma^\dagger \chi \quad (8.17)$$

$$u_{12} = \chi^T S \Sigma \Sigma^\dagger S \chi^* \quad (8.18)$$

As in Eq. (7.25), we may write the Σ - χ potential as a linear combination of these terms,

$$V = \sum_1^{12} \beta_i u_i , \quad (8.19)$$

where the β_i are parameters. The terms $u_1, u_2, u_3, u_7, u_8, u_{11}$ follow from the trivial unitarity of group generators. The terms u_4, u_9 exploit the fact that $\mathbf{7}$ is a real representation, i.e., it is isomorphic to $\bar{\mathbf{7}}$. One finds that, like $SU(2)$, G_2 has only real representations. For the $SU(3)$ model scalar potential, we had

$$\mathbf{2} \sim \epsilon \bar{\mathbf{2}} . \quad (8.20)$$

where $\epsilon = i \sigma^2$. One finds that

$$\mathbf{7} \sim S \bar{\mathbf{7}} , \quad (8.21)$$

where S is the matrix

$$S = \begin{pmatrix} 0 & \mathbb{I}_3 & 0 \\ \mathbb{I}_3 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \quad (8.22)$$

Then the terms u_4, u_{12} are obtained in analogy to the term t_4 from the $SU(3)$ scalar potential. The term u_6 is a slightly more complicated term that combines a $(\mathbf{7}, \mathbf{2})$ with a $(\bar{\mathbf{7}}, \bar{\mathbf{2}})$ using both S and ϵ to form an invariant.

Also, in the $SU(3)$ model, we had a cubic invariant in the potential because it is possible to make a singlet out of three triplets. Similarly, the product of the $\mathbf{7}$ representations also contains a singlet. We note the tensor product decomposition

$$\mathbf{7} \times \mathbf{7} = \mathbf{1} + \mathbf{7} + \mathbf{14} + \mathbf{27}. \quad (8.23)$$

Then we also find

$$\mathbf{7} \times \mathbf{7} \times \mathbf{7} = \mathbf{7} + \mathbf{7} \times \mathbf{7} + \mathbf{7} \times (\mathbf{14} + \mathbf{27}) = \mathbf{1} + \mathbf{7} + \mathbf{7} + \mathbf{14} + \mathbf{27} + \mathbf{7} \times (\mathbf{14} + \mathbf{27}).$$

so that $\mathbf{7}^3$ also contains a singlet. For G_2 we are able to deduce the appropriate Clebsch-Gordon coefficients for constructing the singlet by generalizing the $SU(3)$ coefficients. We find that they are given by a totally antisymmetric tensor C_{ijk} , with

$$C_{714} = C_{725} = C_{736} = 1 \quad , \quad C_{123} = C_{456} = -\sqrt{2} . \quad (8.24)$$

All components that are not related to these five by antisymmetries vanish. The nonzero components of C_{ijk} can be understood in terms of the transformation under the $SU(3)$ subgroup. The terms in the first expression of Eq. (8.24) couple a singlet in the first $\mathbf{7}$ to a $\mathbf{3}$ in the second and a $\bar{\mathbf{3}}$ in the third, forming an $SU(3)$ singlet. The components with indices 123 (456) are just the $SU(3)$ epsilon tensor that couples three $\mathbf{3}$'s ($\bar{\mathbf{3}}$'s) from the three $\mathbf{7}$'s. All other elements of C_{ijk} must be zero since they would not produce an $SU(3)$ invariant. The $\sqrt{2}$ relative normalization between the components is obtained by checking that the term

$$C_{ijk}\psi_i\psi_j\psi_k$$

is invariant under the full action of the group. We check invariance under infinitesimal group elements, which gives the generalization of Eq. (7.24),

$$C_{i'jk}(T^a)_{i'i} + C_{ij'k}(T^a)_{j'j} + C_{ijk}(T^a)_{k'k} = 0. \quad (8.25)$$

We use this equation to compute the relative normalization and confirm that this cubic term is in fact invariant under the group. We use this cubic invariant to form the term u_{10} which combines three G_2 $\mathbf{7}$'s from Σ and χ fields and two $SU(2)$ $\mathbf{2}$'s from Σ to produce an invariant.

Eq. (8.19) is a function with 42 real degrees of freedom, and 12 free parameters. Complete analysis of this potential is beyond the scope of our study, and would involve some intensive methods of nonlinear optimization. Nonetheless, we can show that there are local minima of this potential that produce the desired symmetry-breaking vacuum expectation values. We again assume that the vacuum expectation values take the form given by Eq. (8.1), and minimize the resulting potential over the parameters M, x .

$$V_0 = (2\beta_1 m^2 + \beta_7 x^2 m^2)M^2 - 4\sqrt{2}\beta_{10}xM^3 + (4\beta_2 + 2\beta_3 + \beta_8 x^4 - 2\beta_5)M^4$$

Table 8.1: Spectrum of physical scalars, in units of m , for the example parameter choice described in the text. The states below give a total of 28 real degrees of freedom; the remaining 14 degrees correspond to eaten states.

state	multiplicity	mass	state	multiplicity	mass
$\mathbf{3}_{-1}$	6	4.52	$\mathbf{1}_2$	2	5.24
$\mathbf{3}_0$	3	6.52	$\mathbf{1}_{\pm 1}$	2	2.84
$\mathbf{2}_{\pm 3/2}$	4	4.25	$\mathbf{1}_0$	1	4.03
$\mathbf{2}_{1/2}$	4	3.03	$\mathbf{1}_0$	1	3.61
$\mathbf{2}_{1/2}$	4	2.19	$\mathbf{1}_0$	1	2.25

As an example, for the choice of parameters

$$(\beta_1 \dots \beta_{12}) = (-1, 0.3, 1.1, 1.0, 1.0, 1.0, -1.0, 0.3, 1.0, 1.0, 1.0, 0.1),$$

one finds a global minimum at $M = 2.79m$, $x = 1.25$ and the mass spectrum shown in Table 8.1. The mass spectrum has the correct multiplicities to occupy the representations of uneaten real scalars, and accounts for $42 - 28 = 14$ eaten scalars corresponding to the broken generators of G_2 . The eaten scalars have representations $\mathbf{3}_0, \mathbf{2}_{\pm 3/2}, \mathbf{2}_{\pm 1/2}, \mathbf{1}_{\pm 1}, \mathbf{1}_0$. This parameter choice was random; generally, no fine-tuning is necessary to find solutions. Since we have established that there is no difficulty in finding appropriate symmetry-breaking vacua, we again take M and x as free parameters in the remaining analysis.

8.3 Electroweak Constraints

As with the extended $SU(3)$ model, we can now carry over predictions for shifts to Standard Model electroweak observables by referring to the results of [4] (which were reproduced in Sec. 6.5), and making the identifications

$$c_1 = \frac{c_\phi^4 v^2}{4M^2} \quad \text{and} \quad c_2 = \frac{1}{1 + 2x^2} \frac{c_\psi^4 v^2}{4M^2}.$$

Therefore, the corrections we be identical to those obtained in the extended $SU(3)$ model

The chi-squared analysis for this model is identical to that of the extended $SU(3)$ analysis in Ch. 7. The confidence contours in $c_1 - c_2$ parameter space the same as those given in Fig. 7.5.

The computation of coupling constants and renormalization group equations change substantially due to the new matter content of the G_2 model. The general form of the renormalization group equations is given as

$$\alpha^{-1}(M_U) = \alpha^{-1}(M_Z) + \frac{b_{SM}}{2\pi} \ln \frac{M_U}{M_Z} + \sum_i \frac{b_i}{2\pi} \ln \frac{M_U}{M_i}, \quad (8.26)$$

$$\alpha'^{-1}(M_U) = \alpha'^{-1}(M_Z) + \frac{b'_{SM}}{2\pi} \ln \frac{M_U}{M_Z} + \sum_i \frac{b'_i}{2\pi} \ln \frac{M_U}{M_i} , \quad (8.27)$$

The relevant contributions to beta functions are given in Table 8.2. We follow the same

states	b_i	b'_i
Standard Model	19/6	-41/6
$\mathbf{1}_{\pm 1}$ vector	0	7
$\mathbf{2}_{\pm 1/2}$ vector	7/2	7/2
$\mathbf{2}_{\pm 3/2}$ vector	7/2	63/2
physical scalars	-3/2	-9/2

Table 8.2: Beta functions b_i in Eqs. (8.26) and (8.27). Vector boson beta functions include the contribution from the longitudinal (eaten scalar) component. Physical scalars are assumed to have the same mass as the $\mathbf{2}_{3/2}$ vector bosons.

procedure used for the $SU(3)$ models to construct unification scale contours, boson mass contours, and exclusion regions in $\tilde{g} - \tilde{g}'$ space. The full results and plots are given in an appendix. In this model, the mass of the lightest gauge boson is parameter-dependent. For $x \leq 1$, the $\mathbf{1}_{\pm 1}$ field is the lightest; at $x = 1$ the masses of $\mathbf{1}_{\pm 1}$ and $\mathbf{2}_{\pm 1/2}$ are degenerate; for $x > 1$, the $\mathbf{2}_{\pm 1/2}$ field is the lightest. We note that the $\mathbf{1}_{\pm 1}$ field is always a factor of $\sqrt{2/3}$ lighter than the $\mathbf{2}_{\pm 3/2}$ field, which is the lightest $SU(3)$ field. Therefore, this model has an inherent advantage over the $SU(3)$ models for producing possible observations at particle colliders. We provide plots of the lightest gauge boson at various values of x . The results are qualitatively similar to those for the $SU(3)$ model. Ranges of minimum physically allowed gauge boson masses at approximately optimal choices of x are given in Table 8.3. The states $\mathbf{1}_{\pm 1}$ and $\mathbf{2}_{\pm 1/2}$ both have optimal masses in the 2 TeV range, indicating that they may be good candidates for detection via pair production at the LHC.

8.4 Decays of Exotic Gauge Bosons

In the G_2 model we have excluded the vector-like matter that was used in the extended $SU(3)$ model to facilitate decays of exotic gauge bosons. While the same mechanism could

Table 8.3: Ranges of minimum allowed gauge boson masses in the G_2 model as \tilde{g}' varies from 0.5 to 1.5, for a choice of x near its optimal value. The result in the $SU(3)$ model for the $\mathbf{2}_{\pm 3/2}$ state is the same as in the G_2 model, to the accuracy shown.

State	x	Mass range (TeV)
$\mathbf{2}_{\pm 3/2}$	0.9	2.27 - 2.75
$\mathbf{1}_{\pm 1}$	0.9	1.85 - 2.22
$\mathbf{2}_{\pm 1/2}$	1.5	1.76 - 2.15

be carried over to this model to allow decays of $\mathbf{2}_{\pm 3/2}$, one finds that this approach does not allow decays of the new $\mathbf{1}_{\pm 1}$ and $\mathbf{2}_{\pm 1/2}$ states. Therefore we must search for a different mechanism to allow these decays to avoid the unattractive features of exotic stable, charged matter from the original $SU(3)$ model.

A straightforward way to allow the decays is by introducing a new singlet fermion ν_R , and studying possible operators that arise at a cutoff scale M_F . This allows us to write down the operator responsible for $\mathbf{2}_{\pm 3/2}$ decays,

$$\mathcal{L}_{eff} = \frac{1}{M_F^2} (\bar{e}_R \lambda^{e\dagger} \chi^\dagger) i \not{D} (\Sigma \lambda^\ell \ell_R) + \frac{1}{M_F^2} (\bar{e}_R \lambda^{e\dagger} \chi^\dagger) i \not{D} (\chi \lambda^e e_R) + \text{h.c.} \quad (8.28)$$

The new fermion appears in an effective Lagrangian term as

$$\mathcal{L}_{eff} = \frac{1}{M_F^2} \bar{\nu}_R^c (\chi^T S) \not{D} (\Sigma \ell_L) + \frac{1}{M_F^2} \bar{\nu}_R^c \text{Tr} [\Sigma^T S \not{D} \Sigma \epsilon] e_R^c + \text{h.c.} \quad (8.29)$$

where S, ϵ are the matrices used in the invariant terms for the scalar potential. We then have an effective interaction term

$$\mathcal{L}_{eff} = -\frac{g_2 x}{\sqrt{12}} \left(\frac{M}{M_F} \right)^2 \bar{\nu}_R^c \mathcal{A}_{\mathbf{2}_{1/2}}^i \epsilon_{ij} \ell_L^j - \frac{g_2 x}{\sqrt{3}} \left(\frac{M}{M_F} \right)^2 \bar{\nu}_R^c \mathcal{A}_{\mathbf{1}_{-1}} e_R^c + \text{h.c.} \quad (8.30)$$

The decays vanish in the limits $M_F \rightarrow \infty$ or $x \rightarrow 0$, so the χ field is a necessity for allowing decays. It is worth mentioning that the additional fermion ν_R could provide a dark matter candidate.

8.5 Appendix: Plots of Unification and Gauge Boson Mass Contours

Here we include the full collection of plots for the unification contours and contours of lightest boson mass for the G_2 model. Note that the typesetting on the plots omits tildes from the axes labels. All the plots are actually in $\tilde{g} - \tilde{g}'$ space.

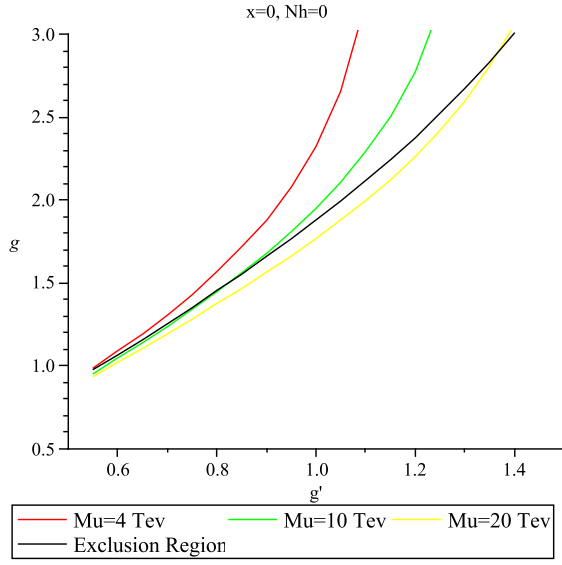


Figure 8.1: Contours of the unification scale M_U for $x = 0$. The region above the black line is excluded at the 95% confidence level.

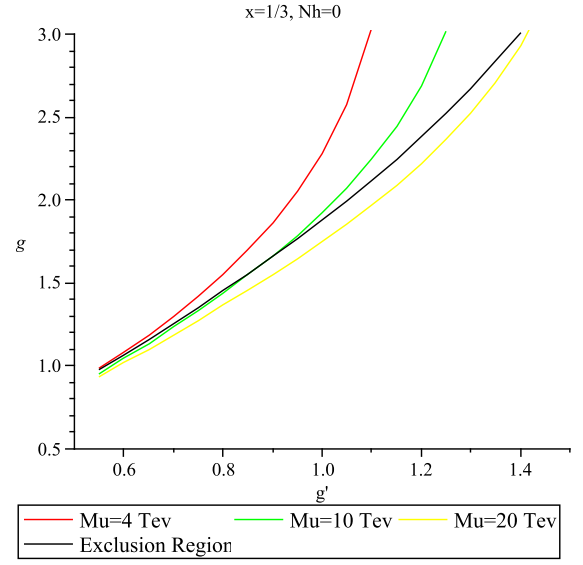


Figure 8.2: Contours of the unification scale M_U for $x = 1/3$. The region above the black line is excluded at the 95% confidence level.

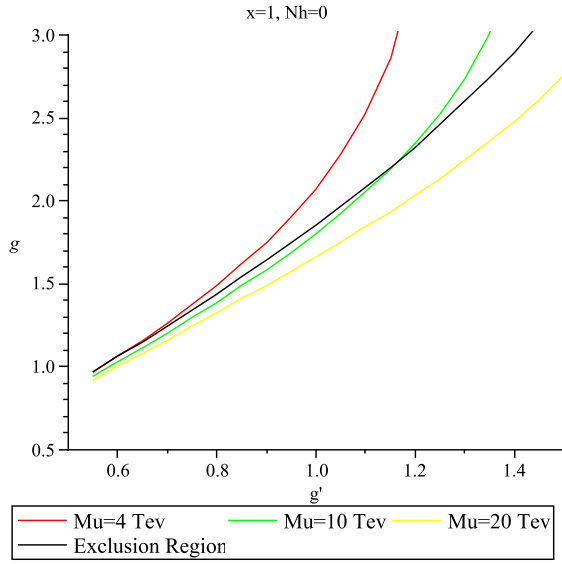


Figure 8.3: Contours of the unification scale M_U for $x = 1$. The region above the black line is excluded at the 95% confidence level.

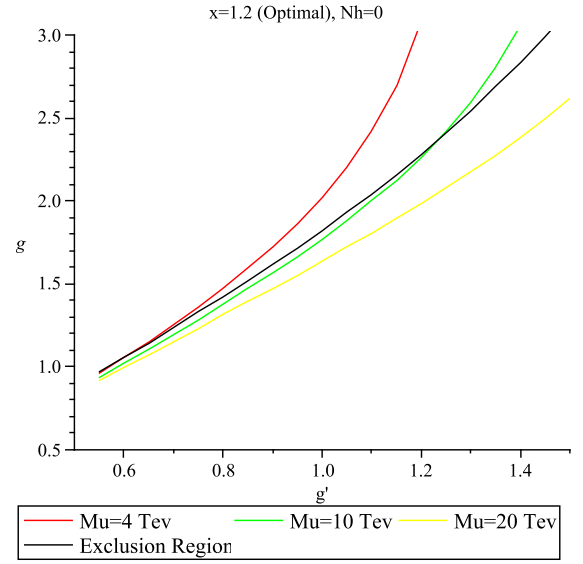


Figure 8.4: Contours of the unification scale M_U for the optimal value of $x = 1.2$. The region above the black line is excluded at the 95% confidence level.

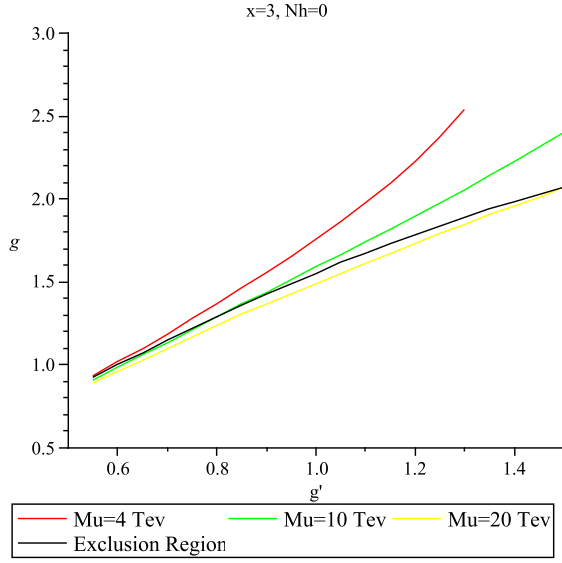


Figure 8.5: Contours of the unification scale M_U for $x = 3$. The region above the black line is excluded at the 95% confidence level.

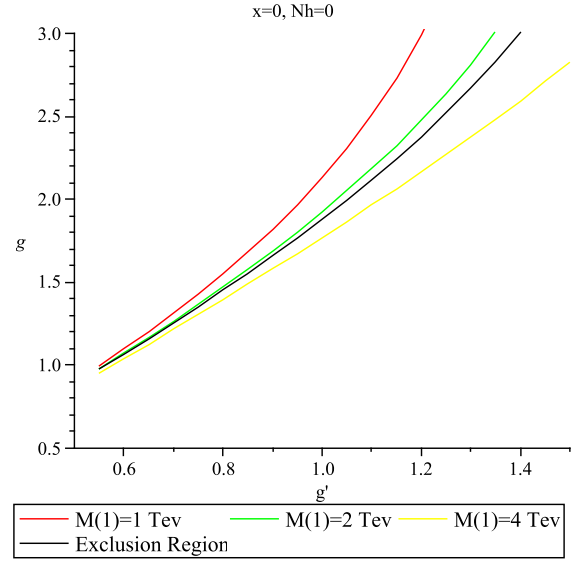


Figure 8.6: Contours of the mass of the lightest gauge boson, for $x = 0$. For this value of x , the lightest boson is $\mathbf{1}_{\pm 1}$. The region above the black line is excluded at the 95% confidence level.

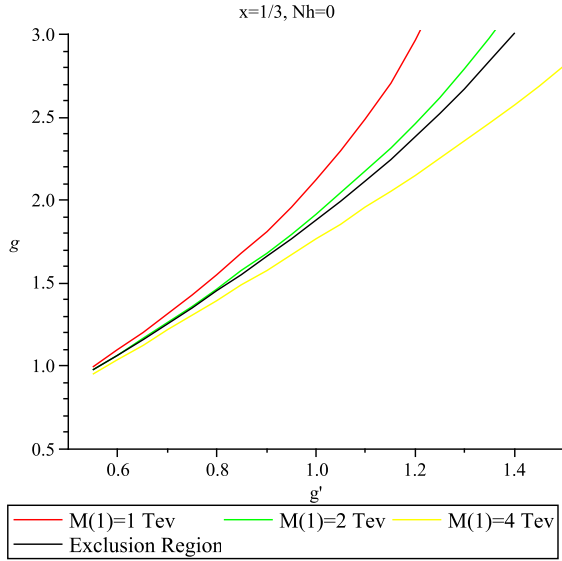


Figure 8.7: Contours of the mass of the lightest gauge boson, for $x = 1/3$. For this value of x , the lightest boson is $\mathbf{1}_{\pm 1}$. The region above the black line is excluded at the 95% confidence level.

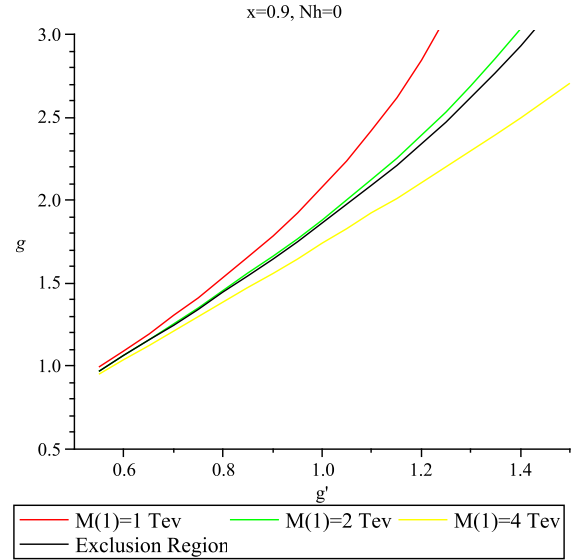


Figure 8.8: Contours of the mass of the lightest gauge boson, for the optimal value of $x = 0.9$. For this value of x , the lightest boson is $\mathbf{1}_{\pm 1}$. The region above the black line is excluded at the 95% confidence level.

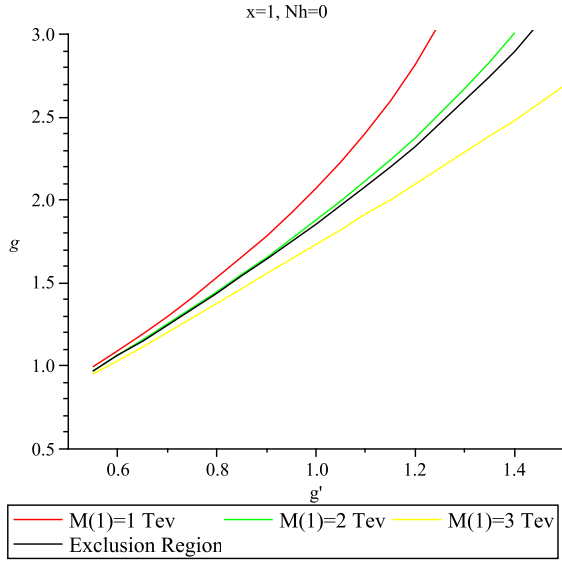


Figure 8.9: Contours of the mass of the lightest gauge boson, for $x = 1$. For this value of x , the $1_{\pm 1}$ and $2_{\pm 1/2}$ bosons have degenerate masses. The region above the black line is excluded at the 95% confidence level.

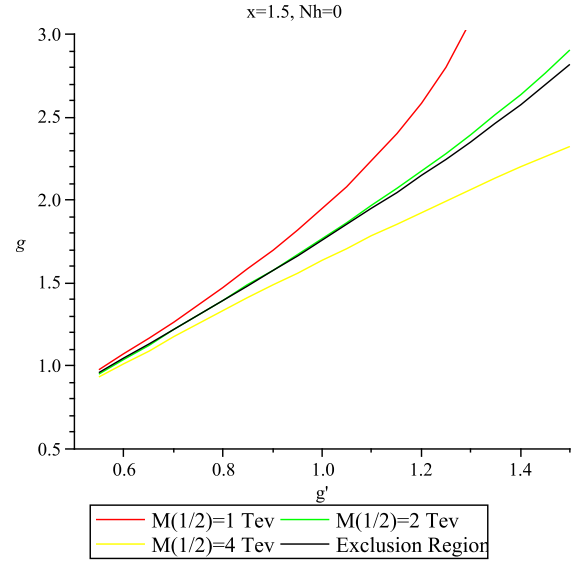


Figure 8.10: Contours of the mass of the lightest gauge boson, for the optimal value $x = 1.5$. For this value of x , the lightest boson is $2_{\pm 1/2}$. The region above the black line is excluded at the 95% confidence level.

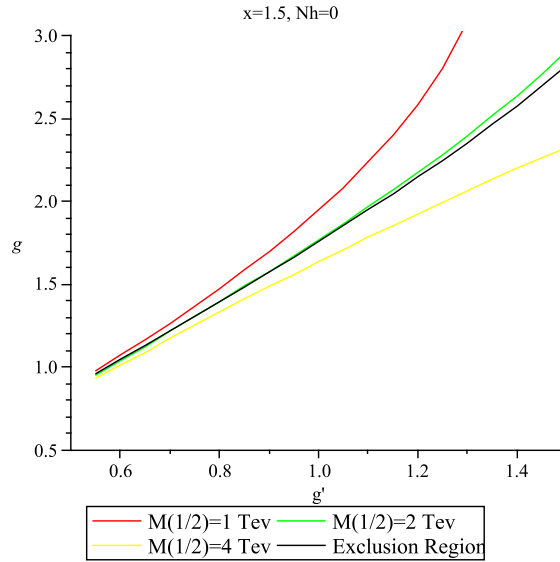


Figure 8.11: Contours of the mass of the lightest gauge boson, for $x = 3$. For this value of x , the lightest boson is $2_{\pm 1/2}$. The region above the black line is excluded at the 95% confidence level.

Chapter 9

Appendix: Orbifold Symmetry Breaking of G_2

In this chapter, we will discuss symmetry-breaking mechanisms that can break the full group G_2 down to Standard Model or other important gauge groups. Here we are interested in orbifold symmetry-breaking. In this approach, we postulate N extra dimensions compactified on a torus (\mathbb{T}^N). If we further introduce some symmetry group H in the extra dimensions, the resulting structure is a \mathbb{T}^N/H orbifold. The group H then produces some symmetry-breaking of G_2 . The elements that are not invariant under action by H are broken, whereas those that are invariant under action by H form the broken subgroup.

For this chapter, we will be interested in the action of the group H on various structures, and so we introduce the notation $h \circ \phi$ to denote the action of $h \in H$ on ϕ . Here we will only use groups H which are cyclic, and therefore correspond to \mathbb{Z}_k . Consequently, when considering group actions of H , we can define the action of the full group by specifying the action

$$\bar{1} \circ \phi = f(\phi),$$

and noting that

$$\bar{\ell} \circ \phi = f^\ell(\phi).$$

Similarly, for a representation P we may define the full representation by specifying $P(\bar{1})$, and deducing that $P(\bar{\ell}) = P(\bar{1})^\ell$.

9.1 Example: $SU(3)$ to $SU(2) \times U(1)$

As a preliminary example, we consider breaking $SU(3)$ to $SU(2) \times U(1)$ on a \mathbb{Z}_2 orbifold. Since we enforcing a \mathbb{Z}_2 symmetry, we need to consider the action of the new symmetry on a field that transforms under the $SU(3)$, and on the $SU(3)$ group elements. We have

$$\begin{aligned} g \circ \phi &= P(h)\phi, \\ g \circ A &= P(h)gP(h)^{-1}, \end{aligned}$$

where P is a representation of the group \mathbb{Z}_2 , ϕ transforms under $SU(3)$, and $g \in SU(3)$. We may fix any representation of \mathbb{Z}_2 of the appropriate dimension; here P must be a 3×3 matrix representation to act on the **3** of $SU(3)$. Define P to be the representation with

$$\begin{aligned} P(\bar{0}) &= \mathbb{I}_3 \\ P(\bar{1}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

To check invariance of group elements, we can simply check the group generators. For an arbitrary element $g \in SU(3)$, $g = e^{iA^a t^a}$, and

$$PgP^{-1} = e^{iPA^a t^a P^{-1}},$$

so a group element is invariant if and only if the generators with nonzero coefficients are invariant. All generators are trivially invariant under the group action of $\bar{0}$ since this is the group identity. We need only to check the action of $\bar{1}$. This action is given schematically by the following matrix

$$\begin{pmatrix} + & + & - \\ + & + & - \\ - & - & + \end{pmatrix};$$

that is, the action of $P(\bar{1})$ is given by entrywise multiplication by this matrix. Entries in the upper 2×2 block and the lower 1×1 block are multiplied by $+1$, and all other entries are multiplied by -1 . This action on the generators of $SU(3)$ gives

$$P(\bar{1})t^a P(\bar{1})^{-1} = \begin{cases} t^a & a = 1, 2, 3, 8 \\ -t^a & a = 4, 5, 6, 7 \end{cases}.$$

Therefore, generators $t^{4\dots 7}$ are broken by the orbifold conditions, and the generators of the $SU(2) \times U(1)$ subgroup remain unbroken.

9.2 G_2 to $SU(2) \times SU(2)$

Using a similar approach, we can break G_2 to $SU(2) \times SU(2)$ on a \mathbb{Z}_2 orbifold. For this analysis, we refer to the basis given by Sec. 5.5. We choose the following representation for \mathbb{Z}_2

$$\begin{aligned} P(\bar{0}) &= \mathbb{I}_7, \\ P(\bar{1}) &= \begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_3 \end{pmatrix}. \end{aligned}$$

Again we need only check the group action of $P(\bar{1})$, for which we have the schematic

$$\left(\begin{array}{c|c} + & - \\ \hline - & + \end{array} \right),$$

where the matrix is partitioned to match $P(\bar{1})$.

$$P(\bar{1})\hat{T}^aP(\bar{1})^{-1} = \begin{cases} \hat{T}^a & a = 1, \dots, 6 \\ -\hat{T}^a & a = 7, \dots, 14 \end{cases}.$$

The generators $\hat{T}^{1\dots 6}$ form the $SU(2) \times SU(2)$ subgroup. Thus we conclude that the G_2 symmetry is broken down to $SU(2) \times SU(2)$.

9.3 $\mathbb{T}^1/\mathbb{Z}_2$ Orbifold

Both symmetry-breaking mechanisms described above involve a \mathbb{Z}_2 orbifold symmetry. This mechanism requires only one extra dimension, so that the resulting orbifold is $\mathbb{T}^1/\mathbb{Z}_2$. In fact, any higher extra dimensional structure will be trivially related to this one-dimensional orbifold by a suitable choice of coordinates, so without loss of generality we only need to consider one extra dimension, which we compactified on a circle of radius R . A point in the 5-dimensional space is specified by (x^μ, y) where x^μ gives the usual 4-dimensional coordinates and y gives the coordinate in the extra dimension. Since the extra dimension is compactified, we identify $y \sim y + n\pi R$. The action of \mathbb{Z}_2 in the 5-dimensional space is given by

$$\bar{1} \circ (x^\mu, y) = (x^\mu, 2\pi R - y) = (x^\mu, -y),$$

in which every point is mapped to its antipode in the extra dimension. Under this action, the points $y = 0, \pi$ are fixed points. The action of \mathbb{Z}_2 on a field in the 5-dimensional space is given by

$$\bar{1} \circ \Phi(x^\mu, y) = \Phi[\bar{1} \circ (x^\mu, y)] = \Phi(x^\mu, -y).$$

The extra-dimensional part of a 5-dimensional wavefunction must respect the periodic boundary conditions in the extra dimension. We can take a Fourier decomposition in the extra dimension, and find that an arbitrary 5-dimensional wavefunction is given by

$$\Phi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \phi^{(n)}(x^\mu) e^{iny/R}.$$

However, we want to expand Φ in the basis formed by eigenstates of the \mathbb{Z}_2 generator $\bar{1}$, which is given by $\{\sin(ny/R), \cos(ny/R)\}$,

$$\begin{aligned} \bar{1} \circ \cos(ny/R) &= \cos(-ny/R) = \cos(ny/R), \\ \bar{1} \circ \sin(ny/R) &= \sin(-ny/R) = -\sin(ny/R). \end{aligned}$$

Thus, the cos states have eigenvalue 1 (even parity), and the sin states have eigenvalue -1 (odd parity). Therefore we can decompose Φ into a sum of even and odd components, as follows

$$\Phi(x^\mu, y) = \Phi^+(x^\mu, y) + \Phi^-(x^\mu, y) = \sum_{n=0}^{\infty} \phi_n^+(x^\mu) \cos(ny/R) + \sum_{n=1}^{\infty} \phi_n^-(x^\mu) \sin(ny/R).$$

9.4 G_2 to $SU(3)$

It is possible to use a \mathbb{Z}_3 orbifold to break G_2 to $SU(3)$; we give an outline of this procedure below. For this procedure it will be necessary to refer to the eigenstate basis of G_2 given by Sec. 5.2. Define Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, the primitive cube root of 1. Choose the following representation of \mathbb{Z}_3

$$\begin{aligned} P(\bar{0}) &= \mathbb{I}_7, \\ P(\bar{1}) &= \begin{pmatrix} \mathbb{I}_3 & 0 & 0 \\ 0 & \omega\mathbb{I}_3 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \\ P(\bar{2}) &= \begin{pmatrix} \mathbb{I}_3 & 0 & 0 \\ 0 & \omega^2\mathbb{I}_3 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \end{aligned}$$

The group representation is of course generated by $P(\bar{1})$, and a matrix will be invariant under action of the group if and only if it is invariant under the action of $P(\bar{1})$. The schematic for this action is

$$\left(\begin{array}{c|c|c} 1 & \omega^2 & \omega \\ \hline \omega & 1 & \omega^2 \\ \hline \omega^2 & \omega & 1 \end{array} \right),$$

where the matrix is partitioned to match $P(\bar{1})$. One can check that

$$P(\bar{1})\tilde{T}^a P(\bar{1})^{-1} = \begin{cases} \tilde{T}^a & a = 1, \dots, 8 \\ \omega\tilde{T}^a & a = 9, \dots, 11 \\ \omega^2\tilde{T}^a & a = 12, \dots, 14. \end{cases}$$

To see this, note that the generators in this basis have the following forms,

$$\tilde{T}^{1\dots 8} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad \tilde{T}^{9\dots 11} = \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}, \quad \tilde{T}^{12\dots 14} = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix}.$$

Since $\tilde{T}^{1\dots 8}$ corresponds to the $SU(3)$ subgroup, we conclude that the \mathbb{Z}_3 breaks the full G_2 symmetry down to $SU(3)$.

9.5 $\mathbb{T}^2/\mathbb{Z}_3$ Orbifold

For the \mathbb{Z}_3 symmetry described above, two extra dimensions are required, so that the resulting orbifold is $\mathbb{T}^2/\mathbb{Z}_3$. Here we use y_1, y_2 to denote the two extra dimensional coordinates, and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. It will be convenient to treat the two extra dimensions as a single complex dimension, so we will make use of the coordinate $z = y_1 + iy_2$.

To obtain the action of \mathbb{Z}_3 , we consider an analogy with the representation of \mathbb{Z}_3 contained on the complex unit circle, i.e. the multiplicative group $\{1, \omega, \omega^2\}$. Here $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, the primitive cube root of unity. The action of the group can be treated by allowing the multiplicative representation to act on the complex coordinate z , or equivalently by considering rotations in $y_1 - y_2$ space by angles $0, 2\pi/3, 4\pi/3$ respectively. The action of the \mathbb{Z}_3 symmetry is given by

$$\bar{1} \circ (x^\mu, z) = (x^\mu, \omega z).$$

Using the matrix representation of complex multiplication or the standard rotation matrices, we may write

$$\bar{1} \circ (x^\mu, \vec{y}) = (x^\mu, \Omega_y \vec{y}),$$

where

$$\Omega_y = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

An immediate problem is that if we take the extra dimensions y_1, y_2 to be orthogonal, the resulting torus is not symmetric under \mathbb{Z}_3 - see Fig. 9.5 for geometric clarification. To correct this, we must define new coordinates ξ_1, ξ_2 . The original y coordinates will be the usual orthogonal coordinates for the extra dimensional space, whereas the new ξ coordinates will be the toroidal coordinates. That is, the ξ coordinates will satisfy the identification $(\xi_1, \xi_2) \sim (\xi_1 + n\pi R, \xi_2 + m\pi R)$. The new structure, a shifted torus, is displayed in Fig. 9.5.

For completeness, we record the coordinate transformations between \vec{y} and $\vec{\xi}$,

$$\vec{y} = \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{3}} \end{pmatrix} \vec{\xi}; \quad \vec{\xi} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \vec{y}.$$

Applying the usual change of basis, we can compute Ω_ξ , the matrix that gives the action of \mathbb{Z}_3 on the ξ coordinates:

$$\bar{1} \circ (x^\mu, \vec{\xi}) = (x^\mu, \Omega_\xi \vec{\xi}).$$

We find

$$\Omega_\xi = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Geometrically, it is easy to identify three fixed points. The first is at $P_0 = (0, 0)$ (unless otherwise specified, we will use the $y_1 - y_2$ coordinates), since the origin is obviously invariant under rotations. The other two occur at the centers of the equilateral triangles that form the torus: $P_1 = \pi R(\frac{1}{2}, \frac{1}{2\sqrt{3}})$, $P_2 = \pi R(0, \frac{1}{\sqrt{3}})$. One can check that

$$\begin{aligned} \bar{1} \circ P_1 &= P_1 - \pi R \hat{\xi}_1 \sim P_1, \\ \bar{1} \circ P_2 &= P_2 - \pi R(\hat{\xi}_1 + \hat{\xi}_2) \sim P_2, \end{aligned}$$

where use $\hat{\xi}_{1,2}$ to indicate the respective unit vectors: $\hat{\xi}_1 = (1, 0)$ and $\hat{\xi}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ in y coordinates.

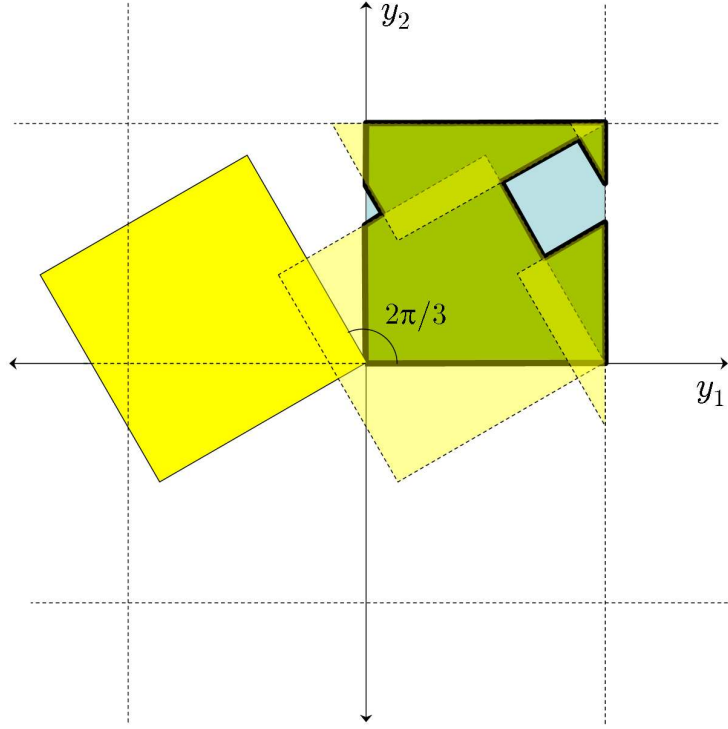


Figure 9.1: The usual torus in coordinates y_1, y_2 is not invariant under the action of \mathbb{Z}_3 , as shown in this diagram. The blue region is the original torus, and the yellow region indicates the result of action by $\bar{1} \in \mathbb{Z}_3$ on the torus (i.e., rotation by $2\pi/3$). The dotted lines and transparent yellow show how this rotated region looks on the original torus.

Again we wish to decompose an arbitrary 6-dimensional wavefunction into eigenstates of the \mathbb{Z}_3 action, where we have

$$\bar{1} \circ \Phi(x^\mu, \vec{\xi}) = \Phi[\bar{1} \circ (x^\mu, \vec{\xi})] = \Phi(x^\mu, \Omega_\xi \vec{\xi}).$$

A two-dimensional Fourier decomposition of Φ gives

$$\Phi(x^\mu, \vec{\xi}) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \phi^{(n,m)}(x^\mu) e^{in\xi_1/R} e^{im\xi_2/R}.$$

We denote $f_{n,m}(\vec{\xi}) = e^{in\xi_1/R} e^{im\xi_2/R}$, which form a set of orthogonal functions with the usual integral inner product. The action of \mathbb{Z}_3 is simply a permutation on the functions $f_{n,m}$. To see this, we consider the action of \mathbb{Z}_3 on $f_{n,m}$,

$$\begin{aligned} \bar{0} \circ f_{n,m}(\xi_1, \xi_2) &= f_{n,m}(\xi_1, \xi_2), \\ \bar{1} \circ f_{n,m}(\xi_1, \xi_2) &= f_{n,m}(-\xi_2, \xi_1 - \xi_2) = f_{m,-n-m}(\xi_1, \xi_2), \\ \bar{2} \circ f_{n,m}(\xi_1, \xi_2) &= f_{n,m}(\xi_2 - \xi_1, -\xi_1) = f_{-n-m,n}(\xi_1, \xi_2). \end{aligned}$$

Every orbit of the permutation has order 3, except for $f_{0,0}$ which is invariant under the action, and therefore has an orbit with order 1. An elementary result concerning permutation

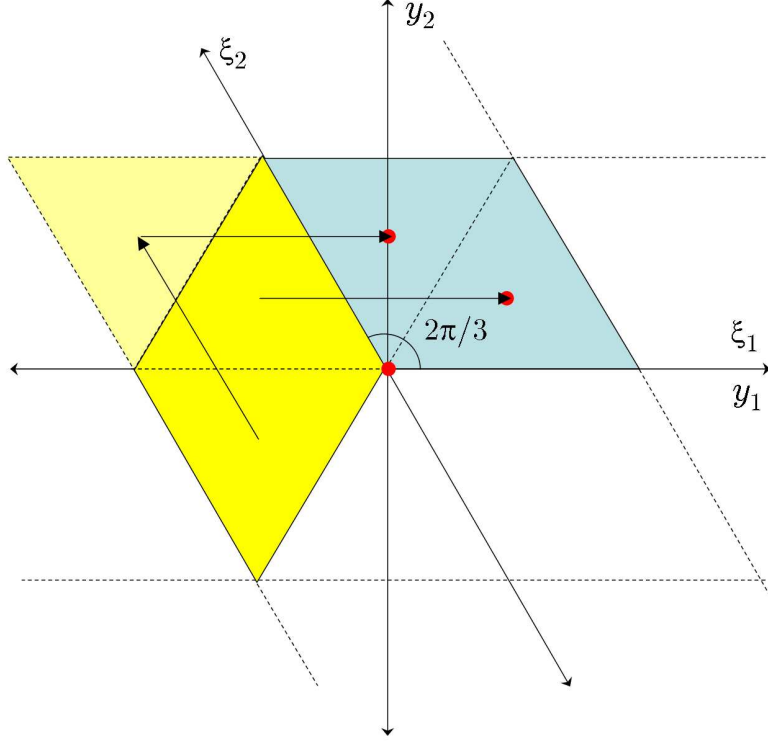


Figure 9.2: The shifted torus in coordinates ξ_1, ξ_2 is invariant under the appropriate action of \mathbb{Z}_3 , as shown in this figure. The blue region is the original torus, and the yellow region indicates the result of action by $\bar{1} \in \mathbb{Z}_3$ on the torus. The dotted arrows and lines indicate how the rotated region is mapped back to the original torus by translations in the compactified space.

operators shows that the eigenvalues such an operation are $1, \omega, \omega^2$. Once the eigenvalues are known, it is straightforward to construct eigenstates. For example, suppose ψ is the extra-dimensional part of some 6-dimensional wavefunction, so that

$$\psi(\vec{\xi}) = \sum_{n,m} c_{n,m} f_{n,m}(\vec{\xi}).$$

Suppose further that ψ is an eigenstate of the \mathbb{Z}_3 action with eigenvalue 1. Then

$$\psi(\vec{\xi}) = \sum_{n,m} c_{n,m} f_{n,m}(\vec{\xi}) = \psi(\Omega_\xi \vec{\xi}) = \sum_{n,m} c_{n,m} f_{m,-n-m}(\vec{\xi}).$$

Due to the orthogonality of the functions $f_{n,m}$, we find that $c_{n,m} = c_{m,-n-m}$. A similar argument considering the action $\vec{\xi} \rightarrow \Omega_\xi^2 \vec{\xi}$ gives the full conclusion that

$$c_{n,m} = c_{m,-n-m} = c_{-n-m,n}.$$

Therefore, any eigenstate associated with eigenvalue 1 can be written as a sum of states $f_{n,m}^1$

$$f_{n,m}^1(\vec{\xi}) = f_{n,m}(\vec{\xi}) + f_{m,-n-m}(\vec{\xi}) + f_{-n-m,n}(\vec{\xi}).$$

Similarly, we have eigenstates of ω, ω^2 respectively

$$\begin{aligned} f_{n,m}^\omega(\vec{\xi}) &= f_{n,m}(\vec{\xi}) + \omega^2 f_{m,-n-m}(\vec{\xi}) + \omega f_{-n-m,n}(\vec{\xi}), \\ f_{n,m}^{\omega^2}(\vec{\xi}) &= f_{n,m}(\vec{\xi}) + \omega f_{m,-n-m}(\vec{\xi}) + \omega^2 f_{-n-m,n}(\vec{\xi}). \end{aligned}$$

The change of basis from $\{f_{n,m}, f_{m,-n-m}, f_{-n-m,n}\}$ to $\{f_{n,m}^1, f_{n,m}^\omega, f_{n,m}^{\omega^2}\}$ is linearly independent, so that the new basis of eigenstates still spans the relevant space of functions on the two extra dimensions. Therefore, we may rewrite an arbitrary 6-dimensional wavefunction in terms of $1, \omega, \omega^2$ eigenstate components

$$\begin{aligned} \Phi(x^\mu, \vec{\xi}) &= \Phi^1(x^\mu, \vec{\xi}) + \Phi^\omega(x^\mu, \vec{\xi}) + \Phi^{\omega^2}(x^\mu, \vec{\xi}) \\ &= \sum_{n,m} \phi_{n,m}^1(x^\mu) f_{n,m}^1(\vec{\xi}) + \sum_{n,m} \phi_{n,m}^\omega(x^\mu) f_{n,m}^\omega(\vec{\xi}) + \sum_{n,m} \phi_{n,m}^{\omega^2}(x^\mu) f_{n,m}^{\omega^2}(\vec{\xi}). \end{aligned}$$

Chapter 10

Conclusions

In this paper we have considered a variety of aspects of the Lie group G_2 . Our construction of representations via root/weight analysis is a completely general procedure, and provides a powerful method for constructing arbitrary representations of any semi-simple Lie group or algebra. Our main motivating goal with G_2 was to develop an electroweak model, extending a minimal $SU(3)$ model. Toward this end, we focused on the $SU(3)$ and $SU(2) \times U(1)$ subgroups of G_2 . We also briefly examined the $SU(2) \times SU(2)$ subgroup, which could be of use in other extensions to the Standard Model.

We then move on to discuss the minimal $SU(3)$ electroweak model, summarizing work done on construction of the model in [3] and on the relevant electroweak constraints in [4]. The minimal $SU(3)$ model suffers from the existence of stable, charged, exotic gauge bosons, a feature which is tightly constrained by empirical observations. This failure motivates two extensions to the minimal $SU(3)$ model, which we analyze in detail.

In the first, we keep the $SU(3) \times SU(2) \times U(1)$ gauge group, and introduce an additional Higgs field and vector-like matter to facilitate decays. We construct and minimize a gauge-invariant scalar potential, and are able to show that there is no fine-tuning necessary to obtain a local minimum with the desired symmetry-breaking vev form and mass spectrum. For this model we repeated the precision electroweak analysis and noted possible experimental signatures of the lightest, possibly long-lived gauge boson. A new free parameter arises from the second Higgs field in this model, and allowed us to relax the constraints on the model slightly.

Finally, we discuss the full G_2 electroweak model, in which we embed the $SU(3)$ into the larger G_2 group. We repeat much of the same analysis, and propose mechanisms for the decay of exotic gauge bosons. We repeat the construction and minimization of a gauge-invariant scalar potential; here the construction involves a non-trivial G_2 triplet invariant. Again, we conclude that no fine-tuning is necessary to obtain a local minimum with the appropriate vev form and mass spectrum. This model is qualitatively similar to the $SU(3)$ model. The main attractive feature of the model is that it includes significantly lighter gauge bosons, making it a more likely candidate for detection at the upcoming collider experiments at the LHC.

In the appendix we summarize some tangential findings concerning the symmetry breaking of G_2 on orbifolds. We originally explored this topic with the intention of embedding the $SU(3)$ group associated with the strong force into G_2 . Difficulties with stable gauge bosons led us to abandon this approach; however the mathematics and geometry of the extra-dimensional symmetry breaking and orbifold are still very interesting, and may be useful in other applications. Furthermore, the analysis of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold may be useful in models and symmetry-breakings involving other groups.

A natural direction for future study is the detailed collider physics of the exotic gauge bosons in these electroweak models. A collider study of the production and detection of the extra gauge bosons in the G_2 model does not exist and is timely given that they may be within the reach of the LHC. The potential for long lifetimes and lepton-flavor violating decays may lead to unique signatures in TeV-scale collider experiments. On the other hand, future work could pursue other extensions to the Standard Model, perhaps involving G_2 or other exceptional groups, using the machinery and preliminary work covered in this thesis.

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